

# On the minimal set of conservation laws and the Hamiltonian structure of the Whitham equations.

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## Abstract

We consider the questions connected with the Hamiltonian properties of the Whitham equations in case of several spatial dimensions. An essential point of our approach here is a connection of the Hamiltonian structure of the Whitham system with the finite-dimensional Poisson bracket defined on the space of periodic or quasi-periodic solutions. From our point of view, this approach gives a possibility to construct the Hamiltonian structure of the Whitham equations under minimal requirements on the properties of the initial system. The Poisson bracket for the Whitham system can be considered here as a deformation of the finite-dimensional bracket with the aid of the Dubrovin - Novikov procedure of bracket averaging. At the end, we consider the examples where the constructions of the paper play an essential role for the construction of the Poisson bracket for the Whitham system.

## 1 Introduction.

In this article we review the questions related to the Hamiltonian formulation of the Whitham averaging method. As is well known, the Whitham method is connected with the slow modulations of periodic or quasiperiodic solutions of partial differential equations (PDE's). Thus, we will consider here systems of PDE's having evolutionary form and the same will be assumed also about the corresponding Whitham system. Besides that, we will assume that the initial evolutionary system has a Hamiltonian structure given by a local field-theoretic Poisson bracket with some local Hamiltonian functional. All our considerations will be made for the  $d$ -dimensional space  $\mathbb{R}^d$  and one time variable  $t$ .

For the systems described above we assume the existence of finite-parametric families of  $m$ -phase periodic or quasiperiodic solutions and consider slow modulations of parameters of these solutions according to the Whitham approach. The main questions considered in the paper will be connected with the construction of the Hamiltonian structure for the Whitham system under some requirements of “completeness” and “regularity” of the corresponding family of  $m$ -phase solutions.

As an intermediate step, we will consider here also the finite-dimensional Poisson brackets, generated by the field-theoretic Hamiltonian structures on the corresponding families of  $m$ -phase solutions. The construction of these brackets will be closely connected with the conservation laws of the initial system, so we introduce here the requirement of existence of a “minimal” set of the conservation laws

for a regular family of  $m$ -phase solutions. As we will see, the same requirement is sufficient also for the construction of the Hamiltonian structure for the corresponding Whitham system.

Our approach in many features will follow the scheme proposed by B.A. Dubrovin and S.P. Novikov and based on the conservative form of the Whitham system. On the other hand, we would like to make here a connection between the finite-dimensional Poisson bracket, given by the restriction of the field-theoretic Poisson bracket on the family of  $m$ -phase solutions, and the Hamiltonian structure of the Whitham system. As we will see, the Hamiltonian structure for the Whitham system can be given as a deformation of the finite-dimensional Poisson bracket with the aid of the Dubrovin - Novikov procedure of averaging of a local Poisson bracket. Let us say that our considerations here will be based just on the minimal requirements on the Hamiltonian properties of the family of  $m$ -phase solutions.

In Chapter 2 we consider in detail the Dirac restriction of the field-theoretic Poisson bracket on the family of  $m$ -phase solutions of a Hamiltonian system. As we will see, many essential features of the construction can be demonstrated here on the example of the KdV hierarchy, so we consider the KdV equation as an illustration of our scheme at the end of the chapter. We give here also some other examples, demonstrating some special cases discussed in the chapter.

In Chapter 3 we discuss the form of a regular Whitham system for a complete Hamiltonian family of  $m$ -phase solutions and the corresponding Hamiltonian structure on the space of slowly modulated parameters.

Chapter 4 is mainly technical in nature and devoted to justification of the construction of the Poisson bracket for the Whitham system.

For the convenience, the technical chapters are provided with a brief description of the content in the beginning of the chapter.

Finally, in Chapter 5 we consider the examples which can be considered as a good illustration of the general scheme presented in the paper. First, we continue here again with the KdV equation and discuss the construction of the Hamiltonian structures for the corresponding Whitham systems. Another example is connected with the equation having just the minimal set of the conservation laws according to our definition.

## 2 Hamiltonian systems and the Poisson brackets on the spaces of $m$ -phase solutions.

In this chapter we will consider the Dirac restriction of a field-theoretic Poisson bracket on a finite-parametric family of  $m$ -phase solutions of a Hamiltonian system.

In general, special features of the bracket restriction depend strongly on the properties of the Hamiltonian operator on the corresponding “submanifold” in the functional space. In our case, the submanifolds of the  $m$ -phase solutions of infinite-dimensional Hamiltonian systems will be characterized by the following common features:

1) A part of the parameters on the family of  $m$ -phase solutions represents the phase (angle) variables changing linearly with time according to the dynamics of the system. The other parameters remain unchanged according to the dynamics of the system.

2) There exists a set of commuting functionals  $\{I^\gamma\}$ , leaving invariant the submanifold of  $m$ -phase solutions according to the infinite-dimensional Poisson structure and generating the linear shifts of the phase variables  $\theta_0^\alpha$  on this submanifold. (The values of the second part of parameters remain

unchanged).

Here we will put some special requirements of “regularity” of the submanifold of  $m$ -phase solutions and the existence of a “minimal” set of commuting functionals  $\{I^\gamma\}$ . In particular, we will assume here that the values of the functionals  $\{I^\gamma\}$  on the family of  $m$ -phase solutions can be naturally chosen as a part of parameters on this family.

As we will see, the main part of the Dirac restriction of the infinite-dimensional Poisson bracket to the submanifold of  $m$ -phase solutions will be connected with the construction of the functionals, representing the phase variables on the family of  $m$ -phase solutions and leaving this family invariant according to the infinite-dimensional Poisson structure. This construction needs in general the resolvability of nontrivial linear systems of PDE's on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions  $\beta_i(\theta^1, \dots, \theta^m)$ . The operators of the corresponding systems are given by the pairwise Poisson brackets of the constraints defining the  $m$ -phase solutions and are closely related with the Hamiltonian operator. As will be shown below, the requirements of “regularity” of the family of  $m$ -phase solutions and the existence of a “minimal” set of commuting functionals  $\{I^\gamma\}$  provide in fact the orthogonality of the right-hand parts of these systems to all the “regular” (left) eigen-vectors of the operators of the systems, corresponding to the zero eigen-value. As a consequence, we can claim in fact the resolvability of these systems in many simple cases, where the spectra of these operators have rather regular form. Thus, under the above requirements we can state in general the possibility of the regular Dirac restriction of an infinite-dimensional Poisson bracket on a submanifold of one-phase solutions of a Hamiltonian system, where the non-zero eigen-values of these operators are in common separated from zero. Other examples of the “regular” situations are connected usually with rather simple Hamiltonian operators, having simple spectral properties both in the single-phase and the multi-phase situations.

In case of possibility of regular Dirac restriction of a Poisson bracket on the submanifold of  $m$ -phase solutions we can introduce the restricted bracket on the space of parameters of the solutions which can in fact be written in rather simple form. Thus, under the requirements of “regularity” of the family of  $m$ -phase solutions and the existence of a “minimal” set of commuting functionals  $\{I^\gamma\}$  the total set of parameters on the family of  $m$ -phase solutions can be chosen in the form

$$(\theta_0^1, \dots, \theta_0^m, \mathbf{k}_1, \dots, \mathbf{k}_d, U^1, \dots, U^{m+s})$$

where  $\mathbf{k}_q = (k_q^1, \dots, k_q^m)$  represent the wave numbers of the solutions and the parameters  $U^\gamma$  coincide with the values of the functionals  $I^\gamma$  on the family. It can be stated then that after an appropriate choice of the initial phase shifts  $\theta_0^\alpha$  the regular Dirac restriction of the infinite-dimensional Poisson bracket on the family of  $m$ -phase solutions can be written in the form

$$\begin{aligned} \{\theta_0^\alpha, \theta_0^\beta\} &= 0, \quad \{\theta_0^\alpha, U^\gamma\} = \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}), \quad \{\theta_0^\alpha, k_p^\beta\} = 0, \\ \{U^\gamma, U^\rho\} &= 0, \quad \{U^\gamma, k_p^\beta\} = 0, \quad \{k_q^\alpha, k_p^\beta\} = 0 \end{aligned} \tag{2.1}$$

where  $\omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$  represent the frequencies, corresponding to the flows generated by the functionals  $I^\gamma$  on the family.

Let us say, however, that the “regular” Dirac restriction of an infinite-dimensional Poisson bracket on the full family of  $m$ -phase solutions of a Hamiltonian system is in fact not possible in general situation. Nevertheless, the bracket (2.1) can still be associated with the infinite-dimensional Poisson bracket of a Hamiltonian system under some additional conditions. Namely, under the same

requirements of “regularity” of the family of  $m$ -phase solutions and the existence of a “minimal” set of commuting functionals (formulated below in detail) we have to require also the possibility of the Dirac restriction of the infinite-dimensional bracket to some special submanifolds which form a dense set in the full family of  $m$ -phase solutions. Let us note here that this requirement is usually rather weak and is satisfied in all examples known to us. At the end of the chapter we give two rather simple examples where two different situations with the Dirac restriction of an infinite-dimensional Poisson bracket to a submanifold of  $m$ -phase solutions naturally arise.

Let us also note here, that in our scheme the bracket (2.1) can be considered in fact as an intermediate step to the bracket for the Whitham system, having more complicated form. However, we must certainly say here that the theory of the finite-dimensional brackets connected with the multi-phase solutions of PDE’s represents also one of the most interesting and important branches of the theory of integrable systems (see [51, 52, 32, 33]).

We will start now more detailed consideration of the Dirac restriction of an infinite-dimensional Poisson bracket on a submanifold of  $m$ -phase solutions of a Hamiltonian system.

In this paper we will consider quasiperiodic  $m$ -phase solutions of Hamiltonian PDE’s having the evolutionary form

$$\varphi_t^i = F^i(\varphi, \varphi_x, \varphi_{xx}, \dots) \equiv F^i(\varphi, \varphi_{x^1}, \dots, \varphi_{x^d}, \dots) \quad (2.2)$$

$i = 1, \dots, n$ ,  $\varphi = (\varphi^1, \dots, \varphi^n)$ , with one time and  $d$  spatial dimensions.

Let us say that we will everywhere define here a quasiperiodic function  $f(\mathbf{x})$  on  $\mathbb{R}^d$  as a function coming from a smooth periodic function of  $m$  variables  $\hat{f}(\boldsymbol{\theta}) = \hat{f}(\theta^1, \dots, \theta^m)$  according to the formula

$$f(x^1, \dots, x^d) = \hat{f}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0)$$

The vectors  $\mathbf{k}_q = (k_q^1, \dots, k_q^m)$  will be called here the wave numbers of the function  $f(\mathbf{x})$ , defining the corresponding mapping  $\mathbb{R}^d \rightarrow \mathbb{T}^m$ . Let us say, that for our purposes here it will be not necessary to put additional requirements on the rational independence of the components of  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ , so our terminology here does not in fact coincide completely with the standard one. Let us, however, keep here this simplified terminology which will not play important role in our considerations. For convenience we will assume here that the function  $\hat{f}(\boldsymbol{\theta})$  is always  $2\pi$ -periodic with respect to each  $\theta^\alpha$ ,  $\alpha = 1, \dots, m$ .

The  $m$ -phase solutions of system (2.2) have the form

$$\varphi^i(\mathbf{x}, t) = \Phi^i(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\omega} t + \boldsymbol{\theta}_0)$$

where the  $2\pi$ -periodic in each  $\theta^\alpha$  functions  $\Phi^i(\boldsymbol{\theta})$  satisfy the system

$$\omega^\alpha \Phi_{\theta^\alpha}^i = F^i\left(\Phi, k_1^{\beta_1} \Phi_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \Phi_{\theta^{\beta_d}}, \dots\right) \quad (2.3)$$

We are going to consider smooth finite parametric families of the quasiperiodic solutions of (2.2) given by the formula

$$\varphi^i(\mathbf{x}, t) = \Phi^i(\mathbf{k}_1(\mathbf{a}) x^1 + \dots + \mathbf{k}_d(\mathbf{a}) x^d + \boldsymbol{\omega}(\mathbf{a}) t + \boldsymbol{\theta}_0, \mathbf{a}) \quad (2.4)$$

with some smooth dependence of the functions  $\Phi^i$  and  $(\mathbf{k}_q, \boldsymbol{\omega})$  on the set of parameters  $\mathbf{a} = (a^1, \dots, a^N)$ . We can see in fact that two different types of parameters naturally arise in the definition (2.4). Thus, the parameters  $\mathbf{a}$  define the “shape” of the solutions  $\varphi^i(\mathbf{x}, t)$  and the corresponding “frequencies” and “wave numbers”. At the same time, the parameters  $\boldsymbol{\theta}_0 = (\theta_0^1, \dots, \theta_0^m)$

represent just the initial phase shifts and take all possible values on the full family of  $m$ -phase solutions.

Let us denote here by  $\Lambda$  the corresponding set of the functions  $\varphi(\mathbf{x})$  in  $\mathbb{R}^d$  given by the formula

$$\varphi^i(\mathbf{x}) = \Phi^i(\mathbf{k}_1(\mathbf{a})x^1 + \dots + \mathbf{k}_d(\mathbf{a})x^d + \boldsymbol{\theta}_0, \mathbf{a}) \quad (2.5)$$

Now we don't put any special requirements on the parameters  $\mathbf{a}$ . We can see according to (2.3) that the family  $\Lambda$  is invariant with respect to the evolutionary system (2.2).

To make a difference between the functions  $\varphi(\mathbf{x})$  and  $\Phi(\boldsymbol{\theta})$  let us denote also by  $\hat{\Lambda}$  the corresponding set of the  $2\pi$ -periodic with respect to each  $\theta^\alpha$  functions  $\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{a})$ , smoothly depending on the parameters  $\mathbf{a} = (a^1, \dots, a^N)$ . Rigorously speaking, we will call here a smooth family of  $m$ -phase solutions of (2.2) a smooth family  $\hat{\Lambda}$  of the functions  $\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{a})$  with the smooth dependence  $\mathbf{k}_q = \mathbf{k}_q(\mathbf{a})$ ,  $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{a})$ , satisfying the corresponding system (2.3).

Let us say now that we will assume here also that system (2.2) is Hamiltonian with respect to some local field-theoretic Poisson bracket given in general by the expression

$$\{\varphi^i(\mathbf{x}), \varphi^j(\mathbf{y})\} = \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\varphi, \varphi_{\mathbf{x}}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d) \quad (2.6)$$

( $l_1, \dots, l_d \geq 0$ ), and has a local Hamiltonian functional of the form

$$H = \int P_H(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \dots) d^d x \quad (2.7)$$

The Hamiltonian structure (2.6) and the Hamiltonian functional (2.7) can be naturally considered on functional spaces of different types. Thus, the structure (2.6) is well defined on the space of rapidly decreasing at infinity functions  $\varphi(\mathbf{x})$ , where (2.7) represents a well defined translationally invariant functional under the appropriate normalization of the density  $P_H$ :  $P_H(0, 0, \dots) = 0$ . Other natural types of the functional spaces can be represented by the spaces of the periodic or the quasiperiodic functions  $\varphi(\mathbf{x})$ . In this case it is natural to define the functional  $H$  in the form

$$H = \lim_{K \rightarrow \infty} \frac{1}{(2K)^d} \int_{-K}^K \dots \int_{-K}^K P_H(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \dots) d^d x$$

Let us note that in the last case we have to define the variation derivative of  $H$  with respect to the variations of  $\varphi(\mathbf{x})$  having the same periodic or quasiperiodic properties as the original function. It's not difficult to see that the standard Euler - Lagrange expressions for the variation derivatives can be used also in this situation.

Here we will write all the local translationally invariant functionals in the general form

$$I = \int P(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \dots) d^d x \quad (2.8)$$

assuming an appropriate definition in every concrete situation.

Let us note also that the Hamiltonian structure (2.6) and the functionals (2.8) can be also naturally considered on the spaces of quasiperiodic functions  $\varphi(\mathbf{x})$  with the fixed wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ , representing natural invariant subspaces for the structure (2.6).

We are going to consider here "maximal" smooth families  $\Lambda$  of  $m$ -phase solutions of system (2.2) in the sense which we will explain below. In particular, we will always assume here that the values

$(\mathbf{k}_1, \dots, \mathbf{k}_d)$  and  $\boldsymbol{\omega}$  represent independent parameters on the family  $\Lambda$ , such that the total set of  $\mathbf{a}$  includes  $N = m(d+1) + s$ , ( $s \geq 0$ ) parameters  $(a^1, \dots, a^N)$ . Thus, we assume in fact that the particular choice of  $\mathbf{a}$  can be represented by the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, n^1, \dots, n^s)$  where  $\mathbf{k}_q$  and  $\boldsymbol{\omega}$  are the wave numbers and the frequencies of the  $m$ -phase solutions and  $\mathbf{n} = (n^1, \dots, n^s)$  are some additional parameters (if any). Including the initial phase shifts  $\boldsymbol{\theta}_0 = (\theta_0^1, \dots, \theta_0^m)$  we then claim that the solutions from the family  $\Lambda$  are parametrized by the  $m(d+1) + s + m$  parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$ .

Let us introduce also the families  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d} \subset \hat{\Lambda}$  consisting of the functions  $\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}) \in \hat{\Lambda}$  with the fixed parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ . The corresponding subsets  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d} \subset \Lambda$  represent families of  $m$ -phase solutions of system (2.2) on the spaces of the quasiperiodic functions with the fixed wave numbers in the coordinate space. The full set of parameters on the families  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  or  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  can be represented by the values  $(\boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$ .

For convenience we will introduce here also the subset  $\mathcal{M} \subset \{\mathbf{k}_1, \dots, \mathbf{k}_d\}$  in the space of parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  given by generic values of  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ , defined by the requirement that the orbits of the group generated by the set of constant vector fields  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  are everywhere dense in  $\mathbb{T}^m$ . It is easy to see that the set  $\mathcal{M}$  has the full measure in the space of the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ .

Let us define the following quasiperiodic functions

$$\begin{aligned}\varphi_{\theta^\alpha}(\mathbf{x}) &= \Phi_{\theta^\alpha}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}) , \\ \varphi_{\omega^\alpha}(\mathbf{x}) &= \Phi_{\omega^\alpha}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}) , \\ \varphi_{n^l}(\mathbf{x}) &= \Phi_{n^l}(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}) ,\end{aligned}$$

$\alpha = 1, \dots, m$ ,  $l = 1, \dots, s$ , on the family  $\Lambda$ .

We will say that the family  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  represents a submanifold in the space of quasiperiodic functions with the fixed wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  if the functions  $(\varphi_{\theta^\alpha}(\mathbf{x}), \varphi_{\omega^\alpha}(\mathbf{x}), \varphi_{n^l}(\mathbf{x}))$  are linearly independent for all values of  $(\boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$ .

It will be convenient here to assume all the properties formulated above whenever we mention the families  $\Lambda$  or  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . So, everywhere below we will assume that the submanifolds  $\Lambda$  and  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  represent the “maximal” families of  $m$ -phase solutions of system (2.2) in the above sense.

### Definition 2.1.

We call the submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  a regular Hamiltonian submanifold in the space of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  if:

1) The bracket (2.6) has on  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  constant number of “annihilators” defined by linearly independent solutions  $\mathbf{v}^{(k)}(\mathbf{x}) = (v_1^{(k)}(\mathbf{x}), \dots, v_n^{(k)}(\mathbf{x}))$ ,  $k = 1, \dots, s'$ , of the equation

$$\sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \Big|_{\Lambda} v_{j, l_1 x^1 \dots l_d x^d}^{(k)}(\mathbf{x}) = 0 \quad (2.9)$$

on the space of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ ;

2) We have:  $m + s \geq s'$ , and:

$$\text{rank} \begin{pmatrix} (\boldsymbol{\varphi}_{\omega^\alpha} \cdot \mathbf{v}^{(k)}) \\ (\boldsymbol{\varphi}_{n^l} \cdot \mathbf{v}^{(k)}) \end{pmatrix} = s'$$

$(\alpha = 1, \dots, m, l = 1, \dots, s, k = 1, \dots, s'),$  where

$$(\varphi_{\omega^\alpha} \cdot \mathbf{v}^{(k)}) \equiv \int \varphi_{\omega^\alpha}^i(\mathbf{x}) v_i^{(k)}(\mathbf{x}) d^d x, \quad (\varphi_{n^l} \cdot \mathbf{v}^{(k)}) \equiv \int \varphi_{n^l}^i(\mathbf{x}) v_i^{(k)}(\mathbf{x}) d^d x$$

are the convolutions of the variation derivatives of annihilators with the corresponding tangent vectors  $\varphi_{\omega^\alpha}, \varphi_{n^l}$ .

We call family  $\Lambda$  a regular Hamiltonian submanifold in the space of quasiperiodic functions if all the families  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  represent regular Hamiltonian submanifolds in the spaces of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  with the same number  $s'$ .

Let us note here that, according to our definition of a quasiperiodic function, Definition 2.1 implies, in particular, that for  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$  the number of smooth linearly independent  $2\pi$ -periodic in each  $\theta^\alpha$  solutions  $\mathbf{v}^{(k)}(\boldsymbol{\theta})$  of the equation

$$\begin{aligned} \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij} (\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{a}), k_1^{\gamma_1}(\mathbf{a}) \Phi_{\theta^{\gamma_1}}, \dots, k_d^{\gamma_d}(\mathbf{a}) \Phi_{\theta^{\gamma_d}}, \dots) \times \\ \times k_1^{\alpha_1^1}(\mathbf{a}) \dots k_1^{\alpha_{l_1}^1}(\mathbf{a}) \dots k_d^{\alpha_1^d}(\mathbf{a}) \dots k_d^{\alpha_{l_d}^d}(\mathbf{a}) v_{j, \theta^{\alpha_1^1} \dots \theta^{\alpha_{l_1}^1} \dots \theta^{\alpha_1^d} \dots \theta^{\alpha_{l_d}^d}}^{(k)} = 0 \end{aligned} \quad (2.10)$$

is exactly equal to  $s'$ .

Let us say also that the smooth periodic solutions of (2.10) define the annihilators with smooth variation derivatives of the Hamiltonian operator

$$\begin{aligned} \hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij} = \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij} (\varphi(\boldsymbol{\theta}), k_1^{\gamma_1} \varphi_{\theta^{\gamma_1}}, \dots, k_d^{\gamma_d} \varphi_{\theta^{\gamma_d}}, \dots) \times \\ \times k_1^{\alpha_1^1} \dots k_1^{\alpha_{l_1}^1} \dots k_d^{\alpha_1^d} \dots k_d^{\alpha_{l_d}^d} \frac{\partial}{\partial \theta^{\alpha_1^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \dots \frac{\partial}{\partial \theta^{\alpha_1^d}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \end{aligned} \quad (2.11)$$

defined on  $\mathbb{T}^m$  for any set  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ .

### Definition 2.2.

We say that a regular Hamiltonian submanifold  $\Lambda$  is equipped with a minimal set of commuting integrals if there exist  $m + s$  functionals  $I^\gamma$ ,  $\gamma = 1, \dots, m + s$ , having the form

$$I^\gamma = \int P^\gamma(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \dots) d^d x \quad (2.12)$$

such that:

1) The functionals  $I^\gamma$  commute with the Hamiltonian (2.7) and with each other with respect to the bracket (2.6):

$$\{I^\gamma, H\} = 0, \quad \{I^\gamma, I^\rho\} = 0, \quad (2.13)$$

2) The values  $U^\gamma$  of the functionals  $I^\gamma$  can be used as the parameters  $\mathbf{a}$  on every submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , such that the total set of parameters on  $\Lambda$  can be represented in the form  $(\mathbf{k}_1, \dots, \mathbf{k}_d, U^1, \dots, U^{m+s}, \boldsymbol{\theta}_0)$ ;

3) The Hamiltonian flows, generated by the functionals  $I^\gamma$ , leave invariant the family  $\Lambda$ , generating the linear time evolution of the phase shifts  $\boldsymbol{\theta}_0$  with constant frequencies  $\boldsymbol{\omega}^\gamma = (\omega^{1\gamma}, \dots, \omega^{m\gamma})$ , such that

$$\text{rk } \|\omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})\| = m$$

everywhere on  $\Lambda$ ;

4) For every submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  the linear space generated by the variation derivatives  $\delta I^\gamma / \delta \varphi^i(\mathbf{x})$  contains the variation derivatives of all the annihilators of bracket (2.6) on the corresponding space of quasiperiodic functions. In other words, on every submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  we can write for some complete set  $\{\mathbf{v}^{(k)}(\mathbf{x})\}$  of linearly independent quasiperiodic solutions of (2.9) the relations:

$$v_i^{(k)}(\mathbf{x}) = \sum_{\gamma=1}^{m+s} \gamma_\gamma^k(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left. \frac{\delta I^\gamma}{\delta \varphi^i(\mathbf{x})} \right|_{\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

with some functions  $\gamma_\gamma^k(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$  on  $\Lambda$ .

Let us also note that in Definition 2.2 we assume in particular that the Jacobian of the coordinate transformation  $(\boldsymbol{\omega}, \mathbf{n}) \rightarrow (U^1, \dots, U^{m+s})$  is different from zero on every  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  whenever we introduce the parameters  $(U^1, \dots, U^{m+s})$  on  $\Lambda$ .

We can see that for any regular Hamiltonian submanifold  $\Lambda$  equipped with a minimal set of commuting integrals we must have the relation  $s = s'$  connecting the number of annihilators of the bracket (2.6) on  $\Lambda$  with the number of the additional parameters  $(n^1, \dots, n^s)$ . This requirement means in fact that the solutions from  $\Lambda$  are parametrized by the set of parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \boldsymbol{\theta}_0)$  on every common level  $N^1 = \text{const}, \dots, N^{s'} = \text{const}$  of the annihilators of the bracket (2.6) on the space of quasiperiodic functions. Let us say that this property presents in most of important examples of  $m$ -phase solutions of Hamiltonian PDE's. So, we will always in fact assume below that  $s = s'$ , such that the number of annihilators of the bracket (2.6) is exactly equal to the number of parameters  $(n^1, \dots, n^s)$ .

It's not difficult to see, that we must have also the relations

$$\sum_{\gamma=1}^{m+s} \gamma_\gamma^k(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \equiv 0 \quad (2.14)$$

for the functions  $\gamma_\gamma^k$  and  $\omega^{\alpha\gamma}$  on any regular Hamiltonian submanifold with a minimal set of commuting integrals.

In the full analogy with the Hamiltonian structure (2.11) we can introduce also the functionals

$$J^\gamma = \int_0^{2\pi} \dots \int_0^{2\pi} P^\gamma \left( \boldsymbol{\varphi}, k_1^{\beta_1} \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots \right) \frac{d^m \theta}{(2\pi)^m}, \quad (2.15)$$

( $\gamma = 1, \dots, m+s$ ), on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions.

Let us note that for any submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  in the space of quasiperiodic functions of  $\mathbf{x}$  we can claim that the corresponding functions  $\Phi_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ ,  $\Phi_{\omega^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ ,  $\Phi_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ , ( $\alpha = 1, \dots, m$ ,  $l = 1, \dots, s$ ), are also linearly independent on  $\mathbb{T}^m$ , being linearly independent on the subset

$$(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0) \mid \text{mod } (2\pi \mathbb{Z})^m \subset \mathbb{T}^m$$

As a result, we can claim that for any submanifold  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  in the space of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  the corresponding family  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  represents a submanifold in the space of smooth functions in  $\mathbb{T}^m$  in the same sense.



In the generic case  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$  the space of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  coincides with the space of smooth functions on  $\mathbb{T}^m$  according to our definition. We can claim also, that the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  corresponds by definition to  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  after the transition to the torus  $\mathbb{T}^m$ . The corresponding Poisson structure (2.6) will be naturally written in this case as

$$\begin{aligned} \{\varphi^i(\boldsymbol{\theta}), \varphi^j(\boldsymbol{\theta}')\} &= \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\boldsymbol{\varphi}(\boldsymbol{\theta}), k_1^{\gamma_1} \boldsymbol{\varphi}_{\theta^{\gamma_1}}, \dots, k_d^{\gamma_d} \boldsymbol{\varphi}_{\theta^{\gamma_d}}, \dots) \times \\ &\times k_1^{\alpha_1^1} \dots k_1^{\alpha_{l_1}^1} \dots k_d^{\alpha_1^d} \dots k_d^{\alpha_{l_d}^d} \frac{\partial}{\partial \theta^{\alpha_1^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \dots \frac{\partial}{\partial \theta^{\alpha_1^d}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \end{aligned} \quad (2.16)$$

while the functionals  $I^\gamma$  will be represented by  $J^\gamma$  after the transition to the functions on  $\mathbb{T}^m$ .

For convenience, let us define here the delta-function  $\delta(\boldsymbol{\theta} - \boldsymbol{\theta}')$  and its higher derivatives  $\delta_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta} - \boldsymbol{\theta}')$  in the  $\boldsymbol{\theta}$ -space by the formula

$$\int_0^{2\pi} \dots \int_0^{2\pi} \delta_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta} - \boldsymbol{\theta}') \psi(\boldsymbol{\theta}') \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \equiv \psi_{\theta^{\alpha_1} \dots \theta^{\alpha_s}}(\boldsymbol{\theta})$$

It will be also convenient here to define the variation derivatives by the rule

$$\delta S \equiv \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta S}{\delta \varphi^i(\boldsymbol{\theta})} \delta \varphi^i(\boldsymbol{\theta}) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m}$$

every time when the integration with respect to  $\boldsymbol{\theta}$  is expected.

For a regular Hamiltonian submanifold  $\Lambda$  equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$  we can claim then, that:

1) The functionals  $J^\gamma$  commute with each other and with the functional

$$J_H = \int_0^{2\pi} \dots \int_0^{2\pi} P_H \left( \boldsymbol{\varphi}, k_1^{\beta_1} \boldsymbol{\varphi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\varphi}_{\theta^{\beta_d}}, \dots \right) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m}$$

according to the Hamiltonian structure (2.16);

2) The values of the functionals  $J^\gamma$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  coincide with the corresponding values of  $I^\gamma$  on the submanifolds  $\Lambda_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ ;

3) The Hamiltonian flows, generated by the functionals  $J^\gamma$  on the space of smooth functions on  $\mathbb{T}^m$  according to bracket (2.16), leave invariant the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , generating the linear evolution of the phase shifts  $\boldsymbol{\theta}_0$  with the same frequencies  $\boldsymbol{\omega}^\gamma(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ .

Indeed, for  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$  all the statements above just follow from the isomorphism of the corresponding (Poisson) spaces of quasiperiodic functions to the space of smooth functions on  $\mathbb{T}^m$  with the Poisson brackets (2.16). Just by continuity we also obtain the same statements for arbitrary  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ . Let us note, however, that in non-generic situation  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \notin \mathcal{M}$  the space of quasiperiodic functions with the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ , defined in our way, does not coincide actually with the space of the smooth functions on  $\mathbb{T}^m$ .

The functions  $\mathbf{v}^{(k)}(\boldsymbol{\theta})$ , given by the relations

$$v_i^{(k)}(\boldsymbol{\theta}) = \sum_{\gamma=1}^{m+s} \gamma_\gamma^k(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left. \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}} \quad (2.17)$$

obviously define annihilators of the corresponding Hamiltonian operators (2.11) on  $\mathbb{T}^m$ . Moreover, for  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$  we can claim that the functions (2.17) define in fact the full set of annihilators of (2.11) with smooth linearly independent variation derivatives on  $\mathbb{T}^m$ . However, it can be easily seen, that in non-generic situation  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \notin \mathcal{M}$  the number of annihilators of the bracket (2.16) increases and in fact is equal to infinity. Thus, we can just claim here that for a regular Hamiltonian submanifold  $\Lambda$  equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$  the values (2.17) define the full set of the “regular” annihilators of the corresponding bracket (2.16), smoothly depending on the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ .

Let us call here the bracket (2.16) the Poisson bracket induced by the bracket (2.6) and the set  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  on  $\mathbb{T}^m$ .

We want to construct now on each submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  a Poisson bracket connected with the Poisson bracket (2.16), defined on the space of all smooth functions on  $\mathbb{T}^m$ . To construct the bracket on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  we are going to use the Dirac procedure of restriction of a Poisson bracket on a submanifold in a functional space. Thus, we will try here to make the Dirac restriction of the bracket (2.16) on the corresponding submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . As a result, we have to obtain a Poisson bracket on the space of parameters  $(U^1, \dots, U^{m+s}, \theta_0^1, \dots, \theta_0^m)$ .

Let us now discuss in detail the procedure of restriction of bracket (2.16) on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . From now on we will assume here that the family  $\Lambda$  represents a regular Hamiltonian submanifold in the space of quasiperiodic functions equipped with a minimal set of commuting integrals  $\{I^1, \dots, I^{m+s}\}$ .

For the Dirac restriction of the bracket (2.16) on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  we need a set of functionals  $\{G^1, \dots, G^{2m+s}\}$  possessing the following properties:

- 1) The values of the functionals  $\{G^1, \dots, G^{2m+s}\}$  represent a coordinate system on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ ;
- 2) The Hamiltonian flows generated by the functionals  $\{G^1, \dots, G^{2m+s}\}$  according to bracket (2.16) leave invariant the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

The Dirac restriction of bracket (2.16) is given then in the coordinate system  $(G^1, \dots, G^{2m+s})$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  just by the pairwise Poisson brackets of the functionals  $\{G^1, \dots, G^{2m+s}\}$ , restricted on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

It is easy to see that the functionals  $\{J^1, \dots, J^{m+s}\}$  can be used as a part of the set  $\{G^1, \dots, G^{2m+s}\}$  giving the coordinates  $(U^1, \dots, U^{m+s})$  on the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . We need, however, to construct also the other part of the set  $\{G^1, \dots, G^{2m+s}\}$ , representing the coordinates  $(\theta_0^1, \dots, \theta_0^m)$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

We have to start with the remark that the values  $\theta_0^\alpha$  represent the cyclic coordinates on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  defined modulo  $2\pi n^\alpha$ ,  $n^\alpha \in \mathbb{Z}$ . So, we have to introduce in fact a set of local maps to define local coordinates on the torus  $\mathbb{T}^m$ . Easy to see that we can cover  $\mathbb{T}^m$  by a set of maps diffeomorphic to  $[0, 1]^m \subset \mathbb{R}^m$  and choose some definite values  $\boldsymbol{\theta}_0 \in \mathbb{R}^m$  in every map among the discrete set defined by the cyclic coordinates on  $\mathbb{T}^m$ .

On the next step we have to define functionals representing the coordinates  $\boldsymbol{\theta}_0$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . According to our remark above, we will actually define them locally in the vicinity of every point  $(\mathbf{U}, \boldsymbol{\theta}_0)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . Let us consider the functionals

$$\vartheta_\alpha = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \varphi^i(\boldsymbol{\theta}) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, J^1, \dots, J^{m+s}) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m}$$

( $\alpha = 1, \dots, m$ ), on the space of smooth functions  $\varphi(\boldsymbol{\theta})$  on  $\mathbb{T}^m$ .

Substituting the functions  $\varphi(\boldsymbol{\theta})$  on  $\mathbb{T}^m$  in the form

$$\varphi^i(\boldsymbol{\theta}) = \Phi^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \quad (2.18)$$

we can see that the values of  $\vartheta_\alpha$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  are equal to zero for  $(\theta_0^1, \dots, \theta_0^m) = (0, \dots, 0)$  while the Jacobian  $|\partial\vartheta_\alpha/\partial\theta_0^\beta|$  is given at  $(\theta_0^1, \dots, \theta_0^m) = (0, \dots, 0)$  by the determinant  $\det ||M_{\alpha\beta}||$ , where

$$M_{\alpha\beta} = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi_{\theta^\beta}^i(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \frac{d^m \theta}{(2\pi)^m}$$

According to the definition of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  we then have

$$\det ||M_{\alpha\beta}|| \neq 0 \quad (2.19)$$

so the mapping

$$(\theta_0^1, \dots, \theta_0^m) \rightarrow \left( \vartheta_1|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \dots, \vartheta_m|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \right) \quad (2.20)$$

is locally invertible.

We can claim then that there exists a constant  $K_{\mathbf{U}} > 0$  such that for the values  $\theta_0^\alpha$ , satisfying the relation

$$-K_{\mathbf{U}} < \theta_0^\alpha < K_{\mathbf{U}} \quad (2.21)$$

the transformations (2.20) are invertible in the neighborhood of the point  $\mathbf{U}$  in the  $\mathbf{U}$ -space, which we consider.

Thus, we can locally write

$$\theta_0^\alpha = \tau^\alpha \left( \vartheta_1|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \dots, \vartheta_m|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \mathbf{U} \right)$$

near the point  $(U^1, \dots, U^{m+s}, 0, \dots, 0)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

As a result, we can say, that the variables  $\theta_0^\alpha$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  can be represented in the vicinity of the point  $(U^1, \dots, U^{m+s}, 0, \dots, 0)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  as the values of the functionals  $\tau^\alpha(\vartheta_1, \dots, \vartheta_m, J^1, \dots, J^{m+s})$  on the functions (2.18).

In the same way, for any point of  $\mathbb{T}^m$ , having coordinates  $(\theta_0^1, \dots, \theta_0^m) = (\zeta^1, \dots, \zeta^m)$  in some local map, we can introduce the functionals

$$\vartheta_\alpha^{[\zeta]} = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \varphi^i(\boldsymbol{\theta}) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta} + \boldsymbol{\zeta}, \mathbf{k}_1, \dots, \mathbf{k}_d, J^1, \dots, J^{m+s}) \frac{d^m \theta}{(2\pi)^m}$$

$(\alpha = 1, \dots, m)$ , and put

$$\theta_0^\alpha = \zeta^\alpha + \tau^\alpha \left( \vartheta_1^{[\zeta]}|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \dots, \vartheta_m^{[\zeta]}|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \mathbf{U} \right)$$

(with the same functions  $\tau^\alpha$ ) near the point  $(U^1, \dots, U^{m+s}, \zeta^1, \dots, \zeta^m)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . Including also the dependence on  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  as the parameters we can locally represent the variables  $\theta_0^\alpha$  as the values of the functionals

$$\theta_0^{\alpha[\zeta]} = \zeta^\alpha + \tau^\alpha \left( \vartheta_1^{[\zeta]}, \dots, \vartheta_m^{[\zeta]}, J^1, \dots, J^{m+s}, \mathbf{k}_1, \dots, \mathbf{k}_d \right)$$

on the corresponding functions from  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  near any point  $(U^1, \dots, U^{m+s}, \zeta^1, \dots, \zeta^m)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

The functionals  $\theta_0^{\alpha[\zeta]}$  give (locally) the corresponding values of  $\theta_0^\alpha$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . In particular, we can write for their Poisson brackets with the functionals  $J^\gamma$ :

$$\left\{ \theta_0^{\alpha[\zeta]}, J^\gamma \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = \omega^{\alpha\gamma}(\mathbf{U})$$

everywhere on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

Let us introduce the “constraints”  $g^{i[\zeta]}(\boldsymbol{\theta})$  defining (locally) the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . We put

$$g^{i[\zeta]}(\boldsymbol{\theta}) = \varphi^i(\boldsymbol{\theta}) - \Phi^i\left(\boldsymbol{\theta} + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{k}_1, \dots, \mathbf{k}_d, J^1, \dots, J^{m+s}\right) \quad (2.22)$$

near the point  $(U^1, \dots, U^{m+s}, \zeta^1, \dots, \zeta^m)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

The functionals  $g^{i[\zeta]}(\boldsymbol{\theta})$  are “numerated” by the index  $i = 1, \dots, n$  and the “continuous index”  $\boldsymbol{\theta} \in \mathbb{T}^m$  and are defined in the same region as the functionals  $\theta_0^{\alpha[\zeta]}$  in the functional space. The equations

$$g^{i[\zeta]}(\boldsymbol{\theta}) = 0$$

define the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  near its “point” with coordinates  $(U^1, \dots, U^{m+s}, \zeta^1, \dots, \zeta^m)$ . Let us denote here by  $\Omega^{[\zeta]} \subset \hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  the corresponding part of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  containing the functions  $\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \in \hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  close to the function  $\Phi(\boldsymbol{\theta} + \boldsymbol{\zeta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ .

The constraints (2.22) are obviously dependent since the following identities take place for the “gradients” of  $g^{i[\zeta]}(\boldsymbol{\theta})$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ :

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \frac{\delta g^{i[\zeta]}(\boldsymbol{\theta})}{\delta \varphi^j(\boldsymbol{\theta}')} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0, \quad \gamma = 1, \dots, m+s \quad (2.23)$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \frac{\delta g^{i[\zeta]}(\boldsymbol{\theta})}{\delta \varphi^j(\boldsymbol{\theta}')} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0, \quad \alpha = 1, \dots, m \quad (2.24)$$

For our purposes we will not need in fact to construct an independent system of constraints and will use system (2.22) everywhere below.

Let us make now one important remark. Namely, we can put in fact one more requirement on the functionals  $\theta_0^{\alpha[\zeta]}$  giving the coordinates  $\theta_0^\alpha$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . Thus, we can actually define the functionals  $\theta_0^{\alpha[\zeta]}$  in such a way, that for the functions

$$h_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) \equiv \frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = \frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \Big|_{\boldsymbol{\varphi} = \Phi[\mathbf{U}, \boldsymbol{\theta}_0]}$$

we have the relations

$$h_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\zeta} + \Delta \boldsymbol{\theta}_0) = h_i^{\alpha[\zeta]}(\boldsymbol{\theta} + \Delta \boldsymbol{\theta}_0; \mathbf{U}, \boldsymbol{\zeta})$$

provided that  $\Phi[\mathbf{U}, \boldsymbol{\zeta} + \Delta \boldsymbol{\theta}_0] \in \Omega^{[\zeta]}$ .

In other words, we can require the relations

$$\frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta} + \Delta \boldsymbol{\theta}_0)} \Big|_{\boldsymbol{\varphi} = \Phi[\mathbf{U}, \boldsymbol{\zeta}]} = \frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \Big|_{\boldsymbol{\varphi} = \Phi[\mathbf{U}, \boldsymbol{\zeta} + \Delta \boldsymbol{\theta}_0]}, \quad (2.25)$$

$\boldsymbol{\theta} \in [0, 2\pi)$ ,  $\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta} + \Delta \boldsymbol{\theta}_0]} \in \Omega^{[\zeta]}$ , where  $\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\theta}_0]} \in \hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  represents the corresponding “point”

$$\varphi(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

Indeed, using the functions  $h_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\zeta})$  we can redefine the functionals  $\theta_0^{\alpha[\zeta]}$  putting

$$\tilde{\theta}_0^{\alpha[\zeta]} = \theta_0^{\alpha[\zeta]} + \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) g^{j[\zeta]}(\boldsymbol{\theta}') \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m}$$

Using the relations

$$\frac{\delta g^{j[\zeta]}(\boldsymbol{\theta}')}{\delta \varphi^i(\boldsymbol{\theta})} = \delta_i^j \delta(\boldsymbol{\theta}' - \boldsymbol{\theta}) - \Phi_{\theta\beta}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{\delta \theta_0^{\beta[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} - \Phi_{U\gamma}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})}$$

we can write

$$\begin{aligned} \left. \frac{\delta \tilde{\theta}_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} &= \left. \frac{\delta \theta_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} + h_i^{\alpha[\zeta]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) - \\ &- \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{\theta\beta}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \left. \frac{\delta \theta_0^{\beta[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} - \\ &- \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{U\gamma}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \left. \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \end{aligned}$$

Using obvious relations

$$\begin{aligned} \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{\theta\beta}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} &= \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}'; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{\theta\beta}^j(\boldsymbol{\theta}' + \boldsymbol{\zeta}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \equiv \delta_\beta^\alpha \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{U\gamma}^j(\boldsymbol{\theta}' + \boldsymbol{\theta}_0^{[\zeta]}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} &= \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} h_j^{\alpha[\zeta]}(\boldsymbol{\theta}'; \mathbf{J}, \boldsymbol{\zeta}) \Phi_{U\gamma}^j(\boldsymbol{\theta}' + \boldsymbol{\zeta}, \mathbf{J}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \equiv 0 \end{aligned}$$

we then easily get

$$\left. \frac{\delta \tilde{\theta}_0^{\alpha[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = h_i^{\alpha[\zeta]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0^{[\zeta]} - \boldsymbol{\zeta}; \mathbf{J}, \boldsymbol{\zeta})$$

From the relations above it is obvious now that the functionals  $\tilde{\theta}_0^{\alpha[\zeta]}$  satisfy the required conditions (2.25). Easy to see also, that the functionals  $\tilde{\theta}_0^{\alpha[\zeta]}$  have exactly the same values on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  as the functionals  $\theta_0^{\alpha[\zeta]}$ .

Let us say, that the choice of the functionals  $\theta_0^{\alpha[\zeta]}$  with the additional invariance property (2.25) is in fact very convenient in many cases. Thus, we will use this choice of  $\theta_0^{\alpha[\zeta]}$  in Chapter 4, where we are going to consider the averaging of the Hamiltonian structures in the Whitham method.

As a particular property of the functionals  $\theta_0^{\alpha[\zeta]}$ , satisfying the requirement (2.25), we can note the fact that their pairwise Poisson brackets on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  do not depend actually on the coordinates  $\theta_0$ . Indeed, according to (2.25) the change of the coordinates  $\theta_0$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  produces just the corresponding shift of  $\theta$  in the “gradients” of  $\theta_0^{\alpha[\zeta]}$ . Easy to see, that the same shift arises also in the coefficients of the operator  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$ , so it will disappear after the integration w.r.t.  $\theta$ . Let us note also, that the functionals  $J^\gamma$  satisfy automatically the requirement (2.25) view their invariance with respect to the transformation  $\theta \rightarrow \theta + \Delta\theta$  on the functional space.

In this chapter the requirement (2.25) will not actually play important role, so we do not impose it here.

Finally, we have to construct the functionals for the coordinates  $\theta_0^\alpha$ , possessing an additional property. Namely, we have to require that the Hamiltonian flows, generated by these functionals, leave invariant the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

We have to modify now the functionals  $\theta_0^{\alpha[\zeta]}$  with the aid of the constraints  $g^{i[\zeta]}(\theta)$  to get the functionals commuting with all  $g^{i[\zeta]}(\theta)$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . Thus, we have to put now

$$\hat{\theta}_0^{\alpha[\zeta]} = \theta_0^{\alpha[\zeta]} + \int_0^{2\pi} \dots \int_0^{2\pi} g^{j[\zeta]}(\theta) \beta_j^{\alpha[\zeta]}(\theta; \mathbf{J}, \theta_0^{\alpha[\zeta]}) \frac{d^m \theta}{(2\pi)^m} \quad (2.26)$$

where the functions  $\beta_i^{\alpha[\zeta]}(\theta; \mathbf{U}, \theta_0)$  are smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions defined by the relations

$$\begin{aligned} \int_0^{2\pi} \dots \int_0^{2\pi} \{g^{i[\zeta]}(\theta), g^{j[\zeta]}(\theta')\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \beta_j^{\alpha[\zeta]}(\theta'; \mathbf{U}, \theta_0) \frac{d^m \theta'}{(2\pi)^m} = \\ = - \left\{ g^{i[\zeta]}(\theta), \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \end{aligned} \quad (2.27)$$

According to (2.27) the functions  $\beta_i^{\alpha[\zeta]}(\theta; \mathbf{U}, \theta_0)$  depend on the coordinates  $(\mathbf{U}, \theta_0)$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  in the vicinity of the point  $(U^1, \dots, U^{m+s}, \zeta^1, \dots, \zeta^m)$  of the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

As we can see, the functions  $\beta_i^{\alpha[\zeta]}(\theta; \mathbf{U}, \theta_0)$  should satisfy at every  $(\mathbf{U}, \theta_0)$  a linear system, where the “matrix” of the system is given by the pairwise Poisson brackets of the constraints  $g^{i[\zeta]}(\theta)$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . It's not difficult to check that the corresponding brackets can be represented in the form

$$\begin{aligned} \{g^{i[\zeta]}(\theta), g^{j[\zeta]}(\theta')\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = & \left\{ \varphi^i(\theta), \varphi^j(\theta') \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} - \\ & - \Phi_{\theta^\beta}^i(\theta + \theta_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ \theta_0^{\beta[\zeta]}, \varphi^j(\theta') \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} - \\ & - \left\{ \varphi^i(\theta), \theta_0^{\gamma[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \Phi_{\theta^\gamma}^j(\theta' + \theta_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) + \\ & + \Phi_{\theta^\beta}^i(\theta + \theta_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ \theta_0^{\beta[\zeta]}, \theta_0^{\gamma[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \Phi_{\theta^\gamma}^j(\theta' + \theta_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \end{aligned} \quad (2.28)$$

In the same way, the right-hand part of system (2.27) can be written as:

$$\begin{aligned}
- \left\{ g^{i[\zeta]}(\boldsymbol{\theta}), \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} &= - \left\{ \varphi^i(\boldsymbol{\theta}), \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} + \\
&+ \Phi_{U^\gamma}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ J^\gamma, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} + \\
&+ \Phi_{\theta^\beta}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ \theta_0^{\beta[\zeta]}, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}
\end{aligned}$$

Let us say, however, that, due to the dependence of the constraints  $g^{i[\zeta]}(\boldsymbol{\theta})$ , system (2.27) can be reduced in fact to a simpler form. Let us prove here the following lemma:

**Lemma 2.1.**

Let the functions  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  satisfy the system

$$\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij} \beta_j^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) = A^{i\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) \quad (2.29)$$

where

$$\begin{aligned}
A^{i\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) &\equiv \\
&\equiv - \left\{ \varphi^i(\boldsymbol{\theta}), \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} + \Phi_{U^\gamma}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ J^\gamma, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}
\end{aligned}$$

Then  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  automatically satisfy system (2.27).

Proof.

Let us first prove the following statement:

Any  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  satisfying system (2.29) automatically satisfy the relations

$$\int_0^{2\pi} \dots \int_0^{2\pi} \Phi_{\theta^\lambda}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad (2.30)$$

$\lambda = 1, \dots, m$ .

Indeed, due to relations (2.23) the right-hand part of system (2.27) is always orthogonal to the variation derivatives  $\delta J^\gamma / \delta \varphi^i(\boldsymbol{\theta})$ ,  $\gamma = 1, \dots, m + s$ . Easy to see that the same property then also takes place for the right-hand part of system (2.29). For any solution of (2.29) this automatically implies the property

$$\omega^{\lambda\gamma}(\mathbf{U}) \int_0^{2\pi} \dots \int_0^{2\pi} \Phi_{\theta^\lambda}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) \frac{d^m \theta}{(2\pi)^m} \equiv 0$$

$\gamma = 1, \dots, m + s$ .

From the part (3) of Definition 2.2 we then immediately get relations (2.30).

Using relations (2.28) for the pairwise brackets of constraints we can now claim that for any  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$ , satisfying system (2.29), the difference between the left- and the right-hand parts of (2.27) is given just by the expression

$$- \Phi_{\theta^\beta}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \int_0^{2\pi} \dots \int_0^{2\pi} \left\{ \theta_0^{\beta[\zeta]}, \varphi^j(\boldsymbol{\theta}') \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \beta_j^{\alpha[\zeta]}(\boldsymbol{\theta}'; \mathbf{U}, \boldsymbol{\theta}_0) \frac{d^m \theta'}{(2\pi)^m} -$$

$$- \Phi_{\theta^\beta}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ \theta_0^{\beta[\zeta]}, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

From the other hand, we know from (2.24) that both the left- and the right-hand parts of (2.27) are orthogonal to the variation derivatives  $\delta\theta_0^{\lambda[\zeta]}/\delta\varphi^i(\boldsymbol{\theta})$ , ( $\lambda = 1, \dots, m$ ), for any  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$ . Applying this property to the above expression we get immediately that the difference between the left- and the right-hand part of system (2.27) is identically equal to zero for any  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  satisfying system (2.29).

Lemma 2.1 is proved.

Let us say that it will be rather convenient to us to choose the solutions  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  of system (2.27) to be also solutions of (2.29) and satisfy the additional property (2.30).

It is easy to see that system (2.29) is resolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions if and only if the system

$$\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij} \tilde{\beta}_j^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) = \Phi_{U^\gamma}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left\{ J^\gamma, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} \quad (2.31)$$

is resolvable on the same space.

Indeed, for any solution  $\tilde{\beta}_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  of system (2.31) we can just put

$$\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) = \tilde{\beta}_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) - \frac{\delta\theta_0^{\alpha[\zeta]}}{\delta\varphi^i(\boldsymbol{\theta})} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

to get a solution of (2.29). Using the expression

$$\left\{ J^\gamma, \theta_0^{\alpha[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = -\omega^{\alpha\gamma}(\mathbf{U})$$

everywhere on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , we can finally formulate the following statement:

System (2.29) is resolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions if and only if the following “test” system

$$\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij} \tilde{\beta}_j^{\alpha}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0) = -\omega^{\alpha\gamma}(\mathbf{U}) \Phi_{U^\gamma}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \quad (2.32)$$

is resolvable on the same space.

Let us note that system (2.32) is well defined globally on the whole submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  and has absolutely identical properties for all  $\boldsymbol{\theta}_0$  under fixed values of  $\mathbf{U}$ . In particular, the spaces of solutions of (2.32) are evidently isomorphic for any two coordinate sets  $(\mathbf{U}, \boldsymbol{\theta}_0)$  and  $(\mathbf{U}, \boldsymbol{\theta}'_0)$ .

Let us give here the following definition:

**Definition 2.3.**

Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of quasiperiodic functions, equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let us fix some values  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ .

We say that the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  admits regular Dirac restriction of bracket (2.16) if the corresponding system (2.32) is resolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions for all values of  $\mathbf{U}$  and has a smooth  $2\pi$ -periodic in each  $\theta^\alpha$  solution  $\tilde{\beta}_i^{\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U})$ , smoothly depending on the parameters  $\mathbf{U}$ .

**Lemma 2.2.**



Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of quasiperiodic functions, equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let system (2.32) have near every value of  $\mathbf{U}$  a smooth  $2\pi$ -periodic in each  $\theta^\alpha$  solution  $\tilde{\beta}_i^\alpha(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U})$ , smoothly depending on the parameters  $\mathbf{U}$ . Then we can construct a global Dirac restriction of the bracket (2.16) on the corresponding submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , having the form:

$$\{\theta_0^\alpha, \theta_0^\beta\} = K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0), \quad \{\theta_0^\alpha, U^\gamma\} = \omega^{\alpha\gamma}(\mathbf{U}), \quad \{U^\gamma, U^\lambda\} = 0 \quad (2.33)$$

with some  $2\pi$ -periodic in each  $\theta^\alpha$  functions  $K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0)$ .

Proof.

As we saw above, the existence of the solution  $\tilde{\beta}_i^\alpha(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U})$  permits to construct the corresponding solutions  $\beta_i^{\alpha[\zeta]}(\boldsymbol{\theta}; \mathbf{U}, \boldsymbol{\theta}_0)$  of system (2.27) and then to define the local functionals  $\hat{\theta}_0^{\alpha[\zeta]}$  introduced in (2.26). The existence of the functionals  $\hat{\theta}_0^{\alpha[\zeta]}$ , together with the set  $(J^1, \dots, J^{m+s})$ , permits to define the Dirac restriction of bracket (2.16) on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , which can be locally written in the form:

$$\begin{aligned} \{\theta_0^\alpha, \theta_0^\beta\}^{[\zeta]} &= \left\{ \hat{\theta}_0^{\alpha[\zeta]}, \hat{\theta}_0^{\beta[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = K^{\alpha\beta[\zeta]}(\mathbf{U}, \boldsymbol{\theta}_0) \\ \{\theta_0^\alpha, U^\gamma\}^{[\zeta]} &= \omega^{\alpha\gamma}(\mathbf{U}), \quad \{U^\gamma, U^\lambda\}^{[\zeta]} = 0 \end{aligned}$$

What we actually have to prove is that any two Dirac restrictions  $\{\dots, \dots\}_D^{[\zeta]}$ ,  $\{\dots, \dots\}_D^{[\zeta']}$ , obtained with the aid of two different sets  $\{\hat{\theta}_0^{\alpha[\zeta]}\}$  and  $\{\hat{\theta}_0^{\alpha[\zeta']}\}$  in two local regions  $\Omega^{[\zeta]}$  and  $\Omega^{[\zeta']}$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , define in fact the same Poisson bracket in the intersection  $\Omega^{[\zeta, \zeta']} = \Omega^{[\zeta]} \cap \Omega^{[\zeta']}$ .

For the proof let us first note that the values of the functionals  $\hat{\theta}_0^{\beta[\zeta]}$  and  $\hat{\theta}_0^{\beta[\zeta']}$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  by construction can differ just by a constant  $2\pi n^\beta$ ,  $n^\beta \in \mathbb{Z}$ , in any connected part of the region  $\Omega^{[\zeta, \zeta']}$ . To prove the Lemma we have to prove then that the Hamiltonian flows, defined by  $\hat{\theta}_0^{\beta[\zeta]}$  and  $\hat{\theta}_0^{\beta[\zeta']}$  on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , coincide in  $\Omega^{[\zeta, \zeta']}$ .

Let us note, that any functional  $\hat{\theta}_0^{\beta[\zeta]} - \hat{\theta}_0^{\beta[\zeta']}$  commutes on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  with all the functionals  $J^\gamma$ ,  $\gamma = 1, \dots, m+s$ , and  $\hat{\theta}_0^{\alpha[\zeta]}$ ,  $\alpha = 1, \dots, m$ , in the region  $\Omega^{[\zeta, \zeta']}$ . Indeed, by construction, the Hamiltonian flows, generated by the functionals  $J^\gamma$  and  $\hat{\theta}_0^{\alpha[\zeta]}$  leave invariant the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , where the functional  $\hat{\theta}_0^{\beta[\zeta]} - \hat{\theta}_0^{\beta[\zeta']}$  has locally constant values. As a result, we can claim, that all the brackets

$$\left\{ J^\gamma, \hat{\theta}_0^{\beta[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \quad \left\{ \hat{\theta}_0^{\alpha[\zeta]}, \hat{\theta}_0^{\beta[\zeta]} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

( $\gamma = 1, \dots, m+s$ ,  $\alpha = 1, \dots, m$ ), coincide with the corresponding brackets

$$\left\{ J^\gamma, \hat{\theta}_0^{\beta[\zeta']} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}, \quad \left\{ \hat{\theta}_0^{\alpha[\zeta]}, \hat{\theta}_0^{\beta[\zeta']} \right\} \Big|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

in the region  $\Omega^{[\zeta, \zeta']}$ . Since the values  $(U^1, \dots, U^{m+s}, \theta_0^1, \dots, \theta_0^m)$  represent a coordinate system on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , we get immediately the required statement.

Considering the values  $(\theta_0^1, \dots, \theta_0^m)$  as the cyclic coordinates on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  we then easily obtain the assertion of the Lemma.

Lemma 2.2 is proved.

Let us give also the proof of the invariance of the restricted bracket with respect to the choice of the functionals  $(I^1, \dots, I^{m+s})$ .

**Lemma 2.3.**

Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of quasiperiodic functions and there exist two different sets of commuting integrals  $(I^1, \dots, I^{m+s})$ ,  $(I'^1, \dots, I'^{m+s})$  satisfying all the requirements of Definition 2.2. Let a submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  satisfy the requirements of Lemma 2.2. Then the Dirac restrictions of bracket (2.16) on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , obtained with the aid of the sets  $(I^1, \dots, I^{m+s})$  and  $(I'^1, \dots, I'^{m+s})$ , coincide with each other.

Proof.

What we have to prove is that the brackets (2.33), obtained with the aid of the sets  $(I^1, \dots, I^{m+s})$  and  $(I'^1, \dots, I'^{m+s})$ , transform into each other under the coordinate transformation

$$(U^1, \dots, U^{m+s}, \theta_0^1, \dots, \theta_0^m) \rightarrow (U'^1, \dots, U'^{m+s}, \theta_0^1, \dots, \theta_0^m)$$

where the coordinates  $\mathbf{U}$  and  $\mathbf{U}'$  are given by the values of  $\mathbf{I}$  and  $\mathbf{I}'$  on the submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ . Easy to see that this requirement is given in our case by the relations

$$\omega'^{\alpha\gamma}(\mathbf{U}) = \frac{\partial U'^\gamma}{\partial U^\rho} \omega^{\alpha\rho}(\mathbf{U})$$

where  $\omega^{\alpha\gamma}(\mathbf{U})$  and  $\omega'^{\alpha\gamma}(\mathbf{U})$  are the frequencies, corresponding to the sets  $\mathbf{I}$  and  $\mathbf{I}'$  respectively.

Consider the sets of the functionals  $J^\gamma$  and  $J'^\gamma$ ,  $\gamma = 1, \dots, m+s$ , introduced with the aid of the sets  $\mathbf{I}$  and  $\mathbf{I}'$  according to formula (2.15). The variation derivatives  $\delta J^\gamma / \delta \varphi^i(\boldsymbol{\theta})$  and  $\delta J'^\gamma / \delta \varphi^i(\boldsymbol{\theta})$  represent regular covectors on the family  $\hat{\Lambda}$ , smoothly depending on all the variables  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}, \boldsymbol{\theta}_0)$ . Besides that, the sets  $\mathbf{J}$  and  $\mathbf{J}'$  generate the same linear space of the vector fields at every point of  $\hat{\Lambda}$  according to bracket (2.16). Since the linear spaces, generated by the gradients of  $\mathbf{J}$  and  $\mathbf{J}'$  contain also all the regular annihilators of the bracket (2.16) on  $\hat{\Lambda}$ , we can claim then, that these spaces coincide with each other at every  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}, \boldsymbol{\theta}_0)$ . According to the translational invariance of the functionals  $\mathbf{J}$  and  $\mathbf{J}'$  we can then write

$$\left. \frac{\delta J'^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = \lambda_\rho^\gamma(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left. \frac{\delta J^\rho}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

on every submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

Easy to see that we naturally have then the relations

$$\omega'^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) = \lambda_\rho^\gamma(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \omega^{\alpha\rho}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

on every  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ , and also

$$\frac{\partial U'^\gamma}{\partial U^\rho} = \lambda_\rho^\gamma(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

according to the definition of the coordinates  $\mathbf{U}$  and  $\mathbf{U}'$ . Thus, we get now the statement of the Lemma.

Lemma 2.3 is proved.

Let us discuss now an analog of the action-angle variables for the restricted bracket.

**Theorem 2.1.**

Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of quasiperiodic functions, equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let system (2.32) have near every value of  $\mathbf{U}$

a smooth  $2\pi$ -periodic in each  $\theta^\alpha$  solution  $\tilde{\beta}_i^\alpha(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U})$ , smoothly depending on the parameters  $\mathbf{U}$ . Then:

1) The Dirac restriction (2.33) of the bracket (2.16) on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  has the form

$$\{\theta_0^\alpha, \theta_0^\beta\} = K^{\alpha\beta}(\mathbf{U}) , \quad \{\theta_0^\alpha, U^\gamma\} = \omega^{\alpha\gamma}(\mathbf{U}) , \quad \{U^\gamma, U^\lambda\} = 0 \quad (2.34)$$

with some skew-symmetric matrix  $K^{\alpha\beta}(\mathbf{U})$ , not depending on  $\boldsymbol{\theta}_0$ ;

2) The relations

$$\{\theta_0^\alpha, \theta_0^\beta\} = 0 , \quad \{\theta_0^\alpha, U^\gamma\} = \omega^{\alpha\gamma}(\mathbf{U}) , \quad \{U^\gamma, U^\lambda\} = 0 \quad (2.35)$$

define a Poisson bracket on the space  $(\mathbf{U}, \boldsymbol{\theta}_0)$ .

3) In the  $\mathbf{U}$ -space there (locally) exists the coordinate transformation

$$Q_\alpha = Q_\alpha(\mathbf{U}) , \quad N^l = N^l(\mathbf{U}) , \quad \bar{\theta}_0^\alpha = \theta_0^\alpha - q^\alpha(\mathbf{U})$$

( $\alpha = 1, \dots, m$ ,  $l = 1, \dots, s$ ), such that the bracket (2.34) takes the form:

$$\begin{aligned} \{\bar{\theta}_0^\alpha, \bar{\theta}_0^\beta\} &= 0 , \quad \{\bar{\theta}_0^\alpha, Q_\beta\} = \delta_\beta^\alpha , \quad \{\bar{\theta}_0^\alpha, N^l\} = 0 , \\ \{Q_\alpha, Q_\alpha\} &= 0 , \quad \{Q_\alpha, N^l\} = 0 , \quad \{N^l, N^p\} = 0 \end{aligned} \quad (2.36)$$

Proof.

Let us consider for bracket (2.33) the Jacobi identities of the form

$$\left\{ \left\{ \theta_0^\alpha, \theta_0^\beta \right\}, U^\gamma \right\} + \left\{ \left\{ \theta_0^\beta, U^\gamma \right\}, \theta_0^\alpha \right\} + \left\{ \left\{ U^\gamma, \theta_0^\alpha \right\}, \theta_0^\beta \right\} \equiv 0$$

We immediately get the relations:

$$\frac{\partial K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0)}{\partial \theta_0^\lambda} \omega^{\lambda\gamma}(\mathbf{U}) \equiv \omega^{\alpha\mu}(\mathbf{U}) \frac{\partial \omega^{\beta\gamma}(\mathbf{U})}{\partial U^\mu} - \omega^{\beta\mu}(\mathbf{U}) \frac{\partial \omega^{\alpha\gamma}(\mathbf{U})}{\partial U^\mu}$$

The right-hand part of the above identity obviously does not depend on  $\boldsymbol{\theta}_0$ , so we get the same for the left-hand part. Since the functions  $K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0)$  are periodic in each  $\theta^\alpha$  we then actually get the relations

$$\frac{\partial K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0)}{\partial \theta_0^\lambda} \omega^{\lambda\gamma}(\mathbf{U}) \equiv 0 \quad (2.37)$$

$$\omega^{\alpha\mu}(\mathbf{U}) \frac{\partial \omega^{\beta\gamma}(\mathbf{U})}{\partial U^\mu} - \omega^{\beta\mu}(\mathbf{U}) \frac{\partial \omega^{\alpha\gamma}(\mathbf{U})}{\partial U^\mu} \equiv 0 \quad (2.38)$$

$\alpha, \beta = 1, \dots, m$ ,  $\gamma = 1, \dots, m + s$ .

According to requirement (3) of Definition 2.2 we then get immediately

$$\frac{\partial K^{\alpha\beta}(\mathbf{U}, \boldsymbol{\theta}_0)}{\partial \theta_0^\lambda} \equiv 0$$

which gives the first part of the Theorem.

It's not difficult to check also that relations (2.38) coincide with the Jacobi identity for the bracket (2.35), so we get also the second part of the Theorem.

To prove the last part of the Theorem, let us note that relations (2.38) express in fact the commutativity of the vector fields

$$\vec{\xi}_{(\alpha)} = (\omega^{\alpha 1}(\mathbf{U}), \dots, \omega^{\alpha m+s}(\mathbf{U}))^t, \quad \alpha = 1, \dots, m$$

on the  $\mathbf{U}$ -space. Since the vector fields  $\vec{\xi}_{(\alpha)}$  are linearly independent, we can then claim that we can locally introduce a coordinate system  $(Q_1, \dots, Q_m, N^1, \dots, N^s)$  in the  $\mathbf{U}$ -space in which the vector fields  $\vec{\xi}_{(\alpha)}$  have the components:

$$\vec{\xi}_{(1)} = (1, 0, \dots, 0)^t, \quad \dots, \quad \vec{\xi}_{(m)} = (0, \dots, 0, 1, 0, \dots, 0)^t \quad (2.39)$$

Easy to see, that relations (2.39) provide then the relations

$$\{\bar{\theta}_0^\alpha, Q_\beta\} = \delta_\beta^\alpha, \quad \{\bar{\theta}_0^\alpha, N^l\} = 0$$

for the Poisson bracket (2.34).

From the Jacobi identities

$$\left\{ \left\{ \theta_0^\alpha, \theta_0^\beta \right\}, \theta_0^\gamma \right\} + \text{c.p.} \equiv 0$$

we have in the new coordinate system for the functions  $K^{\alpha\beta}(\mathbf{Q}, \mathbf{N})$ :

$$\frac{\partial K^{\alpha\beta}(\mathbf{Q}, \mathbf{N})}{\partial Q_\gamma} + \frac{\partial K^{\beta\gamma}(\mathbf{Q}, \mathbf{N})}{\partial Q_\alpha} + \frac{\partial K^{\gamma\alpha}(\mathbf{Q}, \mathbf{N})}{\partial Q_\beta} \equiv 0$$

From the expression above it follows that the functions  $K^{\alpha\beta}(\mathbf{Q}, \mathbf{N})$  can be locally represented in the form

$$K^{\alpha\beta}(\mathbf{Q}, \mathbf{N}) = \frac{\partial q^\beta(\mathbf{Q}, \mathbf{N})}{\partial Q_\alpha} - \frac{\partial q^\alpha(\mathbf{Q}, \mathbf{N})}{\partial Q_\beta}$$

for some smooth functions  $q^\alpha(\mathbf{Q}, \mathbf{N})$ . Putting then  $\bar{\theta}_0^\alpha = \theta_0^\alpha - q^\alpha(\mathbf{Q}, \mathbf{N})$  we get immediately the relations  $\{\bar{\theta}_0^\alpha, \bar{\theta}_0^\beta\} \equiv 0$  for the bracket (2.34).

Theorem 2.1 is proved.

It is natural to call the variables  $Q_\alpha$  the action-type variables and the variables  $\bar{\theta}^\alpha$  - the angle-type variables. The variables  $N^l$  represent the annihilators of the bracket on  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

Let us discuss now the questions of the resolvability of the test system (2.32). First of all, we have to require the orthogonality of the right-hand part of (2.32) to the eigen-vectors of the operator  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$ , corresponding to the zero eigen-value.

It's not difficult to see, that for generic case  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$  this requirement is actually automatically satisfied for system (2.32). Indeed, the kernel vectors of the operator  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  on the space of smooth periodic functions are given in this case by the vectors (2.17), such that we have

$$\begin{aligned} & \sum_{\gamma=1}^{m+s} \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \times \\ & \times \int_0^{2\pi} \dots \int_0^{2\pi} v_i^{(k)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \Phi_{U\gamma}^i(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \frac{d^m \theta}{(2\pi)^m} \equiv \\ & \equiv \sum_{\gamma=1}^{m+s} \gamma_\gamma^k(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \end{aligned}$$

for  $U^\gamma \equiv J^\gamma|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$

The last expression coincides with the left-hand part of (2.14) and is identically equal to zero.

On the other hand, we can see that the analogous property can be definitely violated in the non-generic case  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \notin \mathcal{M}$  where the number of annihilators of  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  (if they exist) can be infinite and is not restricted by the set (2.17).

Easy to see that the simplest situation arises here in the single-phase ( $m = 1$ ) case where system (2.32) is always resolvable. Indeed, all the annihilators of  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  are given in this case by the set (2.17), while the nonzero eigen-values of  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  are separated from zero. It's not difficult to see also that the corresponding solutions of system (2.32) or (2.27) can be chosen here as smooth functions of all the parameters, including the wave numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ . We can then formulate here the following theorem:

**Theorem 2.2.**

*Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of single-phase periodic functions in  $\mathbb{R}^d$ , equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{s+1})$ . Then every submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  admits regular Dirac restriction of the bracket (2.16), smoothly depending on the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ .*

The investigation of resolvability of system (2.32) in the multi-phase ( $m > 1$ ) situation is much more complicated in general. Thus, even under the requirement of orthogonality of the right-hand part of (2.32) to the kernel vectors of the operator  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$ , system (2.32) can still be unresolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions if the eigen-values of  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  are strongly accumulated near the zero value. The properties of the eigen-values of  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  strongly depend in fact on the properties of the numbers  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ , so we have to find in general a subset  $\mathcal{S} \subset \mathcal{M}$  where the corresponding systems (2.32) are resolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions. Easy to see that we can formulate the following theorem:

**Theorem 2.3.**

*Let  $\Lambda$  be a regular Hamiltonian submanifold in the space of quasiperiodic functions in  $\mathbb{R}^d$ , equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let there exist a dense set  $\mathcal{S} \subset \mathcal{M}$  in the space  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  on which the corresponding systems (2.32) satisfy the requirements of Theorem 2.1. Then the relations*

$$\begin{aligned} \{\theta_0^\alpha, \theta_0^\beta\} &= 0, \quad \{\theta_0^\alpha, U^\gamma\} = \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}), \quad \{\theta_0^\alpha, k_p^\beta\} = 0, \\ \{U^\gamma, U^\rho\} &= 0, \quad \{U^\gamma, k_p^\beta\} = 0, \quad \{k_q^\alpha, k_p^\beta\} = 0 \end{aligned} \quad (2.40)$$

*define a Poisson bracket on the space  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}, \boldsymbol{\theta}_0)$ .*

Indeed, it is not difficult to see that the Jacobi identity for bracket (2.40) is given by relations (2.38) for the values

$$\omega^{\alpha\gamma} = \omega^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

which are obviously satisfied under the conditions of the Theorem.

Let us say, however, that the bracket (2.40) can not be obtained in general from the bracket (2.34) after a smooth coordinate transformation in the space  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}, \boldsymbol{\theta}_0)$  since the dependence of the variables  $\bar{\boldsymbol{\theta}}_0$  on the values  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  can have in general rather irregular form.

Let us call here bracket (2.40) the bracket associated with the bracket (2.6) on the family  $\Lambda$ .

As an example, let us consider the Gardner - Zakharov - Faddeev bracket

$$\{\varphi(x), \varphi(y)\} = \delta'(x - y) \quad (2.41)$$

and the KdV equation

$$\varphi_t = \varphi \varphi_x - \varphi_{xxx}$$

corresponding to the Hamiltonian functional

$$H = \int \left( \frac{\varphi^3}{6} + \frac{\varphi_x^2}{2} \right) dx$$

As it is well known ([46]), the KdV equation has a family of  $m$ -phase solutions for any  $m \geq 0$ , which can be represented as the set of extremals of the functionals given by all the linear combinations of the first  $m + 2$  integrals of KdV

$$c_1 \delta I^1 + c_2 \delta I^2 + \dots + c_{m+2} \delta I^{m+2} = 0 \quad (2.42)$$

The first two integrals

$$I^1 = N = \int \varphi dx, \quad I^2 = P = \int \frac{\varphi^2}{2} dx$$

represent here the annihilator and the momentum functional of the bracket (2.41). We have also  $I^3 = H$ , and  $I^k$ ,  $k \geq 4$  represent the higher integrals of the KdV equation.

As was shown in [46], systems (2.42) represent completely integrable finite-dimensional systems having quasiperiodic solutions in the generic case. According to [46], the parameters of the  $m$ -phase solutions are given by  $2m + 1$  real branching points  $(E_1, \dots, E_{2m+1})$ ,  $E_1 < E_2 < \dots < E_{2m+1}$ , of a hyperelliptic surface of genus  $m$  and  $m$  initial phases  $(\theta_0^1, \dots, \theta_0^m)$ . As it is also well-known, the theory of the quasiperiodic solutions of KdV has a remarkable connection with the theory of theta-functions of Riemann surfaces ([11, 12, 27, 28, 13, 14, 15]).

It's not difficult to check that the families  $\Lambda^{(m)}$  represent here regular Hamiltonian submanifolds in the space of quasiperiodic functions, while the functionals  $(I^1, \dots, I^{m+1})$  give a minimal set of commuting integrals for the family  $\Lambda^{(m)}$ .

Easy to see that the operator  $\hat{B}_{\mathbf{k}}$  has here the form

$$\hat{B}_{\mathbf{k}} = k^1 \frac{\partial}{\partial \theta^1} + \dots + k^m \frac{\partial}{\partial \theta^m}$$

Thus, we have to investigate the resolvability of the systems

$$k^1 \tilde{\beta}_{\theta^1}^\alpha + \dots + k^m \tilde{\beta}_{\theta^m}^\alpha = -\omega^{\alpha\gamma}(\mathbf{k}, \mathbf{U}) \Phi_{U^\gamma}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}, \mathbf{U}) \quad (2.43)$$

to define the Dirac restriction of the bracket (2.41) on the submanifolds  $\hat{\Lambda}_{\mathbf{k}}$ .

Let us write the Fourier expansion of the right-hand part of (2.43) in the form:

$$-\omega^{\alpha\gamma}(\mathbf{k}, \mathbf{U}) \Phi_{U^\gamma}(\boldsymbol{\theta}, \mathbf{k}, \mathbf{U}) = \sum_{n_1, \dots, n_m} A_{n_1 \dots n_m}(\mathbf{k}, \mathbf{U}) \exp(in_1 \theta^1 + \dots + in_m \theta^m) \quad (2.44)$$

We have by definition  $U^\gamma = \langle P^\gamma \rangle$ ,  $\gamma = 1, \dots, m+1$ , so we can write the following relations

$$\int_0^{2\pi} \dots \int_0^{2\pi} \Phi_{U^1}(\boldsymbol{\theta}, \mathbf{k}, \mathbf{U}) \frac{d^m \theta}{(2\pi)^m} \equiv 1, \quad \int_0^{2\pi} \dots \int_0^{2\pi} \Phi_{U^\gamma}(\boldsymbol{\theta}, \mathbf{k}, \mathbf{U}) \frac{d^m \theta}{(2\pi)^m} \equiv 0, \quad \gamma \neq 1$$

according to the definition of the functional  $I^1$ . Using the relations  $\omega^{\alpha 1}(\mathbf{k}, \mathbf{U}) \equiv 0$  we then get immediately the relations  $A_{0\dots 0}(\mathbf{k}, \mathbf{U}) \equiv 0$  for the expansion (2.44).

From the theta-functional representation of the right-hand part of (2.43) it is easy to get also that the values  $A_{0\dots 0}(\mathbf{k}, \mathbf{U})$  decay faster than any power of  $|\mathbf{n}|$  at  $|\mathbf{n}| \rightarrow \infty$ , where

$$|\mathbf{n}| \equiv \sqrt{n_1^2 + \dots + n_m^2}$$

To investigate the resolvability of system (2.43) on the set  $(k^1, \dots, k^m) \in \mathcal{M}$  let us define the Diophantine conditions for the values  $(k^1, \dots, k^m)$ . Namely, the vector  $(k^1, \dots, k^m)$  represents a Diophantine vector with the index  $\nu > 0$  and the coefficient  $A > 0$ , if

$$|n_1 k^1(\mathbf{U}) + \dots + n_m k^m(\mathbf{U})| \geq A |\mathbf{n}|^{-\nu}$$

for all  $(n_1, \dots, n_m) \in \mathbb{Z}^m$  ( $(n_1, \dots, n_m) \neq (0, \dots, 0)$ ).

Let us denote here by  $\mathcal{S}_\nu$  the set of all Diophantine vectors  $(k^1, \dots, k^m)$  with index  $\nu$ . The following classical theorem (see e.g. [3, 48]) can be formulated about the space of  $(k^1, \dots, k^m)$ :

*For any  $\nu > m-1$  the measure of the corresponding set of non-Diophantine vectors  $(k^1, \dots, k^m)$  in  $\mathbb{R}^m$  is equal to zero.*

We can see now, that putting  $\mathcal{S} = \mathcal{S}_\nu$  for any  $\nu > m-1$  we get the dense set  $\mathcal{S} \subset \mathcal{M}$  where system (2.43) is resolvable on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions.

According to Theorem 2.3 we can claim now that the corresponding relations (2.40) give a Poisson bracket on the space of parameters  $(\mathbf{k}, \mathbf{U}, \boldsymbol{\theta}_0)$ . It's not difficult to show also that bracket (2.40) gives one of the examples of the analytic Poisson brackets compatible with the KdV theory, introduced by A.P. Veselov and S.P. Novikov ([51, 52]). Let us say, that the methods demonstrated above are applicable in fact for a wide class of Hamiltonian operators (2.6).

Let us note now, that in many examples the restriction of the bracket (2.16) on the submanifolds  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$  can in fact be made in a simpler way than that described above. As an example, let us consider the NLS equation

$$i \psi_t = \psi_{xx} + \kappa |\psi|^2 \psi \quad (2.45)$$

and the Hamiltonian structure

$$\{\psi(x), \bar{\psi}(y)\} = i \delta(x - y) \quad (2.46)$$

As it is well known, the equation (2.45) has the families of  $m$ -phase solutions for any  $m$ , given by the construction, analogous to the KdV case. The bracket (2.46) is non-degenerate and it is easy to check that the full families of  $m$ -phase solutions of NLS represent regular Hamiltonian submanifolds in the space of quasiperiodic functions. Easy to see also that the minimal sets of commuting integrals can be easily constructed here with the aid of the higher integrals of the NLS equation. The operator  $\hat{B}_{\mathbf{k}}^{ij}$  has an ultralocal form in this case and the system (2.32) is trivially solvable for all  $(k^1, \dots, k^m)$ . The Dirac restriction of the bracket (2.16) on the submanifolds  $\hat{\Lambda}_{\mathbf{k}}$  has in this case a regular character and can be always written in the form (2.40) after some smooth change of coordinates  $\theta_0^\alpha \rightarrow \theta_0^\alpha - q^\alpha(\mathbf{k}, \mathbf{U})$ . The relations (2.40) define then a Poisson bracket on the space  $(\mathbf{k}, \mathbf{U}, \boldsymbol{\theta}_0)$

which is always given here by the Dirac restriction of bracket (2.16) on the submanifolds  $\hat{\Lambda}_{\mathbf{k}}$  in the appropriate coordinates.

We have to say here also, that the Poisson and Symplectic structures, defined on the spaces of  $m$ -phase solutions of different systems, play extremely important role in many aspects of the theory of integrable systems (see [51, 52, 33]).

As we will see, bracket (2.35) will play rather important role in the Hamiltonian formulation of the Whitham method, where the parameters  $S^\alpha = \epsilon \theta_0^\alpha$  and  $U^\gamma$  become slow functions of the spatial and time variables:  $S^\alpha \rightarrow S^\alpha(\mathbf{X}, T)$ ,  $U^\gamma \rightarrow U^\gamma(\mathbf{X}, T)$ ,  $\mathbf{X} = \epsilon \mathbf{x}$ ,  $T = \epsilon t$ ,  $\epsilon \rightarrow 0$ . As we will show below, the first two relations in (2.35) should be naturally transformed to the relations

$$\{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\} = 0 ,$$

$$\{S^\alpha(\mathbf{X}), U^\gamma(\mathbf{Y})\} = \omega^{\alpha\gamma}(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y})$$

in this situation. On the other hand, the pairwise Poisson brackets of the functionals  $U^\gamma(\mathbf{X})$ ,  $U^\rho(\mathbf{Y})$  should be defined in this case by a deformation of the bracket (2.35), given by the Dubrovin - Novikov procedure of the bracket averaging. Let us note also, that the values  $k_q^\alpha(\mathbf{X})$  are given here by the relations  $k_q^\alpha(\mathbf{X}) = S_{X^q}^\alpha$  and do not arise as additional parameters.

Let us consider now the Hamiltonian formulation of the Whitham method in more detail.

### 3 Hamiltonian formulation of the Whitham method.

In this chapter we will consider the formulation of the multi-dimensional Whitham method from the Hamiltonian point of view.

As it is well known, in the Whitham method ([53, 54, 55]) we consider “slow-modulated” quasiperiodic solutions of PDE’s, with parameters, slowly depending on the coordinates  $\mathbf{x}$  and  $t$ .

More, precisely, we have to consider asymptotic solutions of system (2.2) with the main term, having the form

$$\varphi_{(0)}^i(\mathbf{x}, t) = \Phi^i\left(\boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_0(\mathbf{X}, T), \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}, T)\right) \quad (3.1)$$

where the functions  $\mathbf{S}(\mathbf{X}, T)$  and  $\mathbf{U}(\mathbf{X}, T)$  are functions of the “slow” variables  $X^q = \epsilon x^q$ ,  $T = \epsilon t$ ,  $\epsilon \rightarrow 0$ .

Following system (2.3), it is easy to see that solution (3.1) satisfies system (2.2) in the main order of  $\epsilon$  under the additional requirement

$$S_T^\alpha = \omega^\alpha(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}, T)) \quad (3.2)$$

However, to avoid growing secular terms in the next corrections to (3.1) we have to put additional requirements also to the functions  $\mathbf{U}(\mathbf{X}, T)$ . Thus, using the substitution

$$\varphi^i(\mathbf{x}, t) \simeq \varphi_{(0)}^i(\mathbf{x}, t) + \epsilon \Psi_{(1)}^i\left(\boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_0(\mathbf{X}, T), \mathbf{X}, T\right) \quad (3.3)$$

we get the linear system

$$\hat{L}_j^i(\mathbf{X}, T) \Psi_{(1)}^j(\boldsymbol{\theta}, \mathbf{X}, T) = f_{(1)}^i(\boldsymbol{\theta}, \mathbf{X}, T) \quad (3.4)$$



where  $\hat{L}_j^i(\mathbf{X}, T) = \hat{L}_{j[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^i$  is the linear operator, given by linearization of system (2.3) on the function

$$\Psi_{(0)}(\boldsymbol{\theta}, \mathbf{X}, T) = \Phi(\boldsymbol{\theta}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}, T))$$

and  $f_{(1)}^i(\boldsymbol{\theta}, \mathbf{X}, T)$  is the first “discrepancy” defined from system (2.2).

To study the resolvability of system (3.4) we have to investigate the properties of the operators  $\hat{L}_{j[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^i$  on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions. If we represent again the parameters  $\mathbf{U}$  in the form:

$$(U^1, \dots, U^{m+s}) = (\omega^1, \dots, \omega^m, n^1, \dots, n^s)$$

we can see that the functions

$$\Phi_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \boldsymbol{\omega}, \mathbf{n}) \quad , \quad \alpha = 1, \dots, m \quad ,$$

$$\Phi_{n^l}(\boldsymbol{\theta}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \boldsymbol{\omega}, \mathbf{n}) \quad , \quad l = 1, \dots, s$$

belong to the kernel of the corresponding operator  $\hat{L}_{j[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^i$ . Let us say that for a good justification of the Whitham method we need in fact the requirement that the vectors  $\Phi_{\theta^\alpha}$ ,  $\Phi_{n^l}$  represent the full set of linearly independent “regular” kernel vectors of the operators  $\hat{L}_{j[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^i$ . Besides that, we have to require also that the operator  $\hat{L}_{j[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^i$  has exactly  $m + s$  regular left eigen-vectors (the eigen-vectors of the adjoint operator), corresponding to the zero eigen-value. More precisely, let us give here the definition of a complete regular family  $\Lambda$  of  $m$ -phase solutions of system (2.2).

**Definition 3.1.**

*Let us call family  $\Lambda$  a complete regular family of  $m$ -phase solutions of system (2.2) if:*

- 1) *The total set of independent parameters on  $\Lambda$  can be represented by the values  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$ ;*
- 2) *The vectors  $\Phi_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ ,  $\Phi_{n^l}(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$  are linearly independent and represent the maximal linearly independent set among the regular kernel vectors of the operator  $\hat{L}_{j[\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}]}^i$ , smoothly depending on the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$  on the whole set of parameters;*
- 3) *The operator  $\hat{L}_{j[\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}]}^i$  has exactly  $m + s$  linearly independent regular left eigen-vectors  $\kappa_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}]}^{(q)}(\boldsymbol{\theta})$ ,  $q = 1, \dots, m + s$ , corresponding to the zero eigenvalue, smoothly depending on the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$  on the whole set of parameters.*

Let us call here the regular Whitham system for a complete regular family of  $m$ -phase solutions of (2.2) the relations (3.2) together with the conditions of orthogonality of the discrepancy  $\mathbf{f}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T)$  to the functions  $\kappa_{[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^{(q)}(\boldsymbol{\theta})$ :

$$\int_0^{2\pi} \dots \int_0^{2\pi} \kappa_{i[\mathbf{S}_\mathbf{X}, \mathbf{U}(\mathbf{X}, T)]}^{(q)}(\boldsymbol{\theta}) f_{(1)}^i(\boldsymbol{\theta}, \mathbf{X}, T) \frac{d^m \theta}{(2\pi)^m} = 0 \quad , \quad q = 1, \dots, m + s \quad (3.5)$$

In the single-phase case conditions (3.5) provide the resolvability of system (3.4) at every  $\mathbf{X}$  and  $T$  on the space of  $2\pi$ -periodic in  $\theta$  functions. Moreover, in the case  $m = 1$  the corresponding asymptotic solution  $\varphi(\theta, \mathbf{X}, T, \epsilon)$  of system (2.2) can be usually represented in the form of the regular asymptotic series:

$$\varphi^i(\theta, \mathbf{X}, T, \epsilon) = \sum_{k \geq 0} \epsilon^k \Psi_{(k)}^i \left( \frac{S(\mathbf{X}, T)}{\epsilon} + \theta, \mathbf{X}, T \right)$$

(see e.g. [34]).

Certainly, the situation is much more complicated in the multi-phase ( $m > 1$ ) case, where the behavior of the eigen-values of  $\hat{L}_j^i(\mathbf{X}, T)$  can be highly nontrivial. As a rule, the study of the corrections to the main approximation (3.1) requires in this case rather complicated mathematical methods. Let us give here the references on the papers [6, 7, 8] where the investigation of the multi-phase case can be found. As follows from the results of [6, 7, 8], the correction to the main term (3.1) behaves here in more complicated way than that represented in (3.3), however, it also vanishes at  $\epsilon \rightarrow 0$  under the fulfillment of conditions (3.5).

In general, we can claim, that the regular Whitham system (3.2), (3.5) plays the central role in the description of the slow modulations both in the single-phase and the multi-phase situations.

It is well known in the Whitham approach that the Whitham system does not give any restrictions on the functions  $\theta_0(\mathbf{X}, T)$  and is connected just with the functions  $\mathbf{k}_q(\mathbf{X}, T) = \mathbf{S}_{X^q}$  and  $\mathbf{U}(\mathbf{X}, T)$  (see e.g. [53, 54, 55, 34]). A simple proof of this statement for system (3.5) under the assumptions, formulated above, can be found in [40, 41]. It can be also shown under some assumptions that the phase shifts  $\theta_0^\alpha(\mathbf{X}, T)$  can in fact be absorbed by the functions  $S^\alpha(\mathbf{X}, T)$  after an appropriate correction of initial data (see e.g. [24, 25, 38, 9]). At the same time, the corresponding initial phase shift can still play rather important role in consideration of the slowly modulated solutions in the so-called weakly nonlinear case ([44], see also [39, 9]). Let us represent here also just some incomplete list of the classical papers devoted to the foundations of the Whitham method: [1, 4, 5, 6, 7, 8, 16, 18, 19, 20, 23, 26, 30, 32, 34, 44, 45, 47, 53, 54, 55].

In the rest of this paper we are going to consider the Hamiltonian properties of the regular Whitham system, given by (3.2) and (3.5). As we will see, the corresponding Hamiltonian structure will be connected to some extent with the structures, considered in the previous chapter.

The Hamiltonian theory of the Whitham equations was started in the pioneer works of B.A. Dubrovin and S.P. Novikov ([16, 17, 18, 19]). In the approach of B.A. Dubrovin and S.P. Novikov the Whitham system was considered as a system of Hydrodynamic Type. For the case of one spatial dimension systems of this kind can be written in the following general form:

$$U_T^\nu = V_\mu^\nu(\mathbf{U}) U_X^\mu, \quad \nu, \mu = 1, \dots, N \quad (3.6)$$

The general Dubrovin - Novikov bracket for system (3.6) can be written in the form

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(\mathbf{U}(X)) \delta'(X - Y) + b_\lambda^{\nu\mu}(\mathbf{U}(X)) U_X^\lambda \delta(X - Y) \quad (3.7)$$

Theory of brackets (3.7) is closely connected with the Differential Geometry. Thus, expression (3.7) with non-degenerate tensor  $g^{\nu\mu}(\mathbf{U})$  defines a Poisson bracket on the space of fields  $\mathbf{U}(X)$  if and only if the values  $g^{\nu\mu}(\mathbf{U})$  represent a flat contravariant pseudo-Riemannian metric on the space of  $\mathbf{U}$ , while the values  $\Gamma_{\nu\lambda}^\mu(\mathbf{U}) = -g_{\nu\tau}(\mathbf{U}) b_\lambda^{\tau\mu}(\mathbf{U})$  coincide with the corresponding Christoffel symbols  $(g_{\nu\tau}(\mathbf{U}) g^{\tau\mu}(\mathbf{U}) \equiv \delta_\nu^\mu)$ .

The Hamiltonian properties of systems (3.6) are also closely related with their integrability. Thus, according to conjecture of S.P. Novikov, any diagonalizable system (3.6), which is Hamiltonian with respect to some bracket (3.7), can be integrated. The Novikov conjecture was proved by S.P. Tsarev ([49, 50]), who suggested a method for solving diagonalizable Hamiltonian systems (3.6). In fact, the method of Tsarev is applicable to a wider class of diagonalizable systems (3.6). The corresponding class of systems (3.6) was called by S.P. Tsarev semi-Hamiltonian and includes also the systems, Hamiltonian with respect to the weakly nonlocal generalizations of the Dubrovin - Novikov bracket - the Mokhov - Ferapontov bracket ([43]) and general Ferapontov brackets ([21, 22]).

The Dubrovin - Novikov procedure of averaging of a Poisson bracket is closely connected with the conservative form of the Whitham system in the case of one spatial dimension ( $d = 1$ ). The initial system has in this case the form

$$\varphi_t^i = F^i(\varphi, \varphi_x, \dots) \quad (3.8)$$

and is supposed to be Hamiltonian with respect to a local Poisson bracket

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{(k)}^{ij}(\varphi, \varphi_x, \dots) \delta^{(k)}(x - y)$$

with a local Hamiltonian functional

$$H = \int P_H(\varphi, \varphi_x, \dots) dx$$

The Dubrovin - Novikov procedure is based on the existence of  $N = 2m + s$  local integrals

$$I^\nu = \int P^\nu(\varphi, \varphi_x, \dots) dx$$

commuting with the Hamiltonian  $H$  and with each other

$$\{I^\nu, H\} = 0, \quad \{I^\nu, I^\mu\} = 0$$

For the time evolution of the densities  $P^\nu(x)$  we can then write:

$$P_t^\nu(\varphi, \varphi_x, \dots) \equiv Q_x^\nu(\varphi, \varphi_x, \dots)$$

for some functions  $Q^\nu(\varphi, \varphi_x, \dots)$ .

In the same way, the pairwise Poisson brackets of the densities  $P^\nu(x)$ ,  $P^\mu(y)$  can be represented in the form:

$$\{P^\nu(x), P^\mu(y)\} = \sum_{k \geq 0} A_k^{\nu\mu}(\varphi, \varphi_x, \dots) \delta^{(k)}(x - y)$$

where

$$A_0^{\nu\mu}(\varphi, \varphi_x, \dots) \equiv \partial_x Q^{\nu\mu}(\varphi, \varphi_x, \dots)$$

for some functions  $Q^{\nu\mu}(\varphi, \varphi_x, \dots)$ .

The number of the functionals  $I^\nu$  is equal here to the number of the parameters  $(k^1, \dots, k^m, \omega^1, \dots, \omega^m, n^1, \dots, n^s)$  and the full regular Whitham system can be written in the conservative form

$$\langle P^\nu \rangle_T = \langle Q^\nu \rangle_X, \quad \nu = 1, \dots, N, \quad (3.9)$$

where the operation  $\langle \dots \rangle$  means the averaging on the family of  $m$ -phase solutions of (3.8):

$$\langle f \rangle = \int_0^{2\pi} \dots \int_0^{2\pi} f(\Phi, k^\alpha \Phi_{\theta^\alpha}, \dots) \frac{d^m \theta}{(2\pi)^m}$$

Using the parameters  $U^\nu = \langle P^\nu \rangle$  instead of the parameters  $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$  on  $\Lambda$ , we can write the averaged Poisson bracket on the space of the functions  $\mathbf{U}(\mathbf{X})$  in the form ([16, 17, 18, 19]):

$$\{U^\nu(X), U^\mu(Y)\} = \langle A_1^{\nu\mu} \rangle(\mathbf{U}) \delta'(X - Y) + \frac{\partial \langle Q^{\nu\mu} \rangle}{\partial U^\gamma} U_X^\gamma \delta(X - Y) \quad (3.10)$$

At the same time, the averaging of the Hamiltonian functional  $H$  gives the Hamiltonian functional

$$H_{av} = \int_{-\infty}^{+\infty} \langle P_H \rangle (\mathbf{U}(X)) dX$$

for the Whitham system (3.9).

In paper [36] the proof of the Jacobi identity for bracket (3.10) under certain regularity assumptions about the family of  $m$ -phase solutions of system (3.8) was suggested. Let us say also, that in paper [35] the consistency of the bracket averaging procedure with the averaging of the Lagrangian structure in the case when both the procedures are possible was also established. The most detailed consideration of the justification of the averaging of local field-theoretic Hamiltonian structures in one-dimensional case can be found in [40], where the new results concerning the appearance of “resonances” on the space of  $m$ -phase solutions of (3.8) are also presented. Let us also note, that the generalization of the Dubrovin - Novikov procedure for the weakly nonlocal Poisson brackets was also suggested in [37].

We have to say, however, that the form (3.6) of the Whitham system is not so convenient in the multi-dimensional situation because of the presence of additional constraints in this case. Besides that, the requirement of the presence of a complete set of integrals  $I^\nu$ ,  $\nu = 1, \dots, m(d+1) + s$ , seems to be too strict in the case  $d > 1$  and is not satisfied in a number of examples. Instead, it is more convenient to consider the Whitham system in the form (3.2), (3.5), where the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  are given by the derivatives of the function  $\mathbf{S}(\mathbf{X}, T)$ . As a result, it is also more natural to represent the corresponding Hamiltonian structure in the form, considered in the previous chapter. Thus, we will separate here the “phase” variables  $(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}))$  and the “density” variables  $(U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$ . As we will see, the procedure of the averaging of a Poisson bracket can be made in this case just in presence of a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$  on the family of  $m$ -phase solutions of (2.2). Unfortunately, the procedure described in the paper still can not be considered as a general method for the multi-dimensional ( $d \geq 2$ ) integrable PDE’s since both the Hamiltonian structures and the conservation laws usually have there more complicated form. However, the scheme we present here seems to be rather useful in many interesting examples.

Let us note also here that in paper [42] the possibility of reduction of the necessary number of commuting integrals was considered from another point of view, using the concept of “pseudo-phases”, first introduced by Whitham ([55]). The approach, used in [42] is applicable mostly to special physical systems, having some additional symmetry properties.

Thus, we put again now  $d \geq 1$  and consider  $m$ -phase solutions of system (2.2), which is supposed to be Hamiltonian with respect to the multi-dimensional Poisson bracket (2.6) with the Hamiltonian functional (2.7).

We will also assume here that the family  $\Lambda$  represents a complete regular family of  $m$ -phase solutions of system (2.2) and there exist  $m + s$  functionals (2.12), such that their values can be used as a full set of parameters  $\mathbf{U}$  at every fixed values of  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ .

From the conditions

$$\text{rk} \left\| \frac{\partial U^\gamma}{\partial \omega^\alpha} \quad \frac{\partial U^\gamma}{\partial n^l} \right\| = m + s$$

we easily get the following statement:

1) The vectors

$$\{\Phi_{\omega^\alpha}(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}), \Phi_{n^l}(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})\}$$

( $\alpha = 1, \dots, m$ ,  $l = 1, \dots, s$ ), are linearly independent at every  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ ;

2) The covectors

$$\zeta_{i[\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}]}^{(\gamma)}(\boldsymbol{\theta}) = \left. \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})} \quad (3.11)$$

( $\gamma = 1, \dots, m + s$ ), are linearly independent at every  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ ;

3)

$$\text{rk } \|(\zeta^{(\gamma)} \cdot \Phi_{\omega^\alpha}) \quad (\zeta^{(\gamma)} \cdot \Phi_{n^l})\| = m + s$$

at every  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ , where  $(\cdot \cdot)$  means the standard convolution

$$(\zeta \cdot \xi) \equiv \int_0^{2\pi} \dots \int_0^{2\pi} \zeta_i(\boldsymbol{\theta}) \xi^i(\boldsymbol{\theta}) \frac{d^m \theta}{(2\pi)^m}$$

on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions.

It is not difficult to check also, that for any translationally invariant first integral  $I^\gamma$  of system (2.2), having the form (2.12), the corresponding covector  $\zeta_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}]}^{(\gamma)}(\boldsymbol{\theta})$  represents a regular left eigen-vector of the operator  $\hat{L}_{j[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}]}^i$ , corresponding to the zero eigen-value. Thus, we can formulate here the following proposition:

**Proposition 3.1.**

*Let the family  $\Lambda$  represent a complete regular family of  $m$ -phase solutions of system (2.2) and there exist  $m + s$  first integrals (2.12), such that their values can be used as a full set of parameters  $\mathbf{U}$  at every fixed values of  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ . Then the corresponding covectors  $\zeta_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}]}^{(\gamma)}(\boldsymbol{\theta})$ , defined by (3.11), generate the full space of the regular left eigen-vectors of the operators  $\hat{L}_{j[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}]}^i$ , corresponding to the zero eigen-value.*

For consideration of the Hamiltonian structure it will be convenient to write now the regular Whitham system in a slightly different form.

**Lemma 3.1.**

*Let the family  $\Lambda$  represent a complete regular family of  $m$ -phase solutions of system (2.2) and there exist  $m + s$  first integrals (2.12), such that their values can be used as a full set of parameters  $\mathbf{U}$  at every fixed values of  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ . Let the time evolution of the corresponding densities  $P^\gamma(\mathbf{x})$  according to system (2.2) have the form*

$$P_t^\gamma(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \boldsymbol{\varphi}_{xx}, \dots) = Q_{x^1}^{\gamma 1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \boldsymbol{\varphi}_{xx}, \dots) + \dots + Q_{x^d}^{\gamma d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \boldsymbol{\varphi}_{xx}, \dots)$$

*with some functions  $(Q^{\gamma 1}(\boldsymbol{\varphi}, \dots), \dots, Q^{\gamma d}(\boldsymbol{\varphi}, \dots))$ .*

*Then the system*

$$S_T^\alpha = \omega^\alpha(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}) \quad , \quad \alpha = 1, \dots, m, \quad (3.12)$$

$$U_T^\gamma = \langle Q^{\gamma 1} \rangle_{X^1} + \dots + \langle Q^{\gamma d} \rangle_{X^d} \quad , \quad \gamma = 1, \dots, m + s$$

*( $U^\gamma \equiv \langle P^\gamma \rangle$ ), is equivalent to system (3.2), (3.5).*

Proof.

We have actually to prove the equivalence of the second part of system (3.12) to relations (3.5) under conditions (3.2).

For convenience, let us introduce the functions

$$\Pi_i^{\gamma(l_1 \dots l_d)}(\varphi, \varphi_{\mathbf{x}}, \dots) \equiv \frac{\partial P^\gamma(\varphi, \varphi_{\mathbf{x}}, \dots)}{\partial \varphi_{l_1 x^1 \dots l_d x^d}} \quad , \quad l_1, \dots, l_d \geq 0$$

From the expression for the evolution of the densities  $P^\gamma(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots)$  we then get the following identities:

$$\begin{aligned} \sum_{l_1, \dots, l_d} \epsilon^{l_1 + \dots + l_d} \Pi_i^{\gamma(l_1 \dots l_d)}(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots) (F^i(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots))_{l_1 X^1 \dots l_d X^d} &\equiv \\ &\equiv \epsilon Q_{X^1}^{\nu_1}(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots) + \dots + \epsilon Q_{X^d}^{\nu_d}(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots) \end{aligned} \quad (3.13)$$

Let us substitute now the functions

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) = \Phi^i \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}) \right) \quad (3.14)$$

in the identity (3.13). Easy to see that every operator  $\epsilon \partial / \partial X^p$  acting on functions (3.14) can be naturally represented as the sum of the term  $S_{X^p}^\alpha \partial / \partial \theta^\alpha$  and the terms, proportional to  $\epsilon$ . So, for any function  $f(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots)$  on the submanifold (3.14) we can write in fact the expansion

$$f(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots) = \sum_{l \geq 0} \epsilon^l f_{(l)} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{X} \right)$$

where all  $f_{(l)}(\boldsymbol{\theta}, \mathbf{X})$  represent some local functions of  $(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}))$  and their derivatives.

It's not difficult to see also, that we can write:

$$\begin{aligned} \langle Q^{\gamma_1} \rangle_{X^1} + \dots + \langle Q^{\gamma_d} \rangle_{X^d} &= \int_0^{2\pi} \dots \int_0^{2\pi} \left( Q_{X^1(1)}^{\gamma_1} + \dots + Q_{X^d(1)}^{\gamma_d} \right) \frac{d^m \theta}{(2\pi)^m} = \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left( \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} F_{l_1 X^1 \dots l_d X^d(1)}^i + \Pi_{i(1)}^{\gamma(l_1 \dots l_d)} F_{l_1 X^1 \dots l_d X^d(0)}^i \right) \frac{d^m \theta}{(2\pi)^m} = \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left( \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} S_{X^1}^{\gamma_1^1} \dots S_{X^1}^{\gamma_{l_1}^1} \dots S_{X^d}^{\gamma_1^d} \dots S_{X^d}^{\gamma_{l_d}^d} F_{(1) \theta^{\gamma_1^1} \dots \theta^{\gamma_{l_1}^1} \dots \theta^{\gamma_1^d} \dots \theta^{\gamma_{l_d}^d}}^i + \right. \\ &\quad \left. + \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} (\omega^\beta \Phi_{\theta^\beta}^i)_{l_1 X^1 \dots l_d X^d(1)} + \Pi_{i(1)}^{\gamma(l_1 \dots l_d)} (\omega^\beta \Phi_{\theta^\beta}^i)_{l_1 X^1 \dots l_d X^d(0)} \right) \frac{d^m \theta}{(2\pi)^m} = \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \left( S_{X^1}^{\gamma_1^1} \dots S_{X^1}^{\gamma_{l_1}^1} \dots S_{X^d}^{\gamma_1^d} \dots S_{X^d}^{\gamma_{l_d}^d} (-1)^{l_1 + \dots + l_d} \Pi_{i(0) \theta^{\gamma_1^1} \dots \theta^{\gamma_{l_1}^1} \dots \theta^{\gamma_1^d} \dots \theta^{\gamma_{l_d}^d}}^{\gamma(l_1 \dots l_d)} F_{(1)}^i + \right. \\ &\quad \left. + \omega_{X^1}^\beta \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} l_1 \Phi_{\theta^\beta}^i{}_{(l_1-1) X^1 \dots l_d X^d(0)} + \dots + \omega_{X^d}^\beta \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} l_d \Phi_{\theta^\beta}^i{}_{l_1 X^1 \dots (l_d-1) X^d(0)} + \right. \\ &\quad \left. + \omega^\beta \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} \Phi_{\theta^\beta}^i{}_{l_1 X^1 \dots l_d X^d(1)} + \omega^\beta \Pi_{i(1)}^{\gamma(l_1 \dots l_d)} \Phi_{\theta^\beta}^i{}_{l_1 X^1 \dots l_d X^d(0)} \right) \frac{d^m \theta}{(2\pi)^m} \end{aligned}$$

Let us note now that the last two terms in the expression above represent the integral of the value

$$\omega^\beta \sum_{l_1, \dots, l_d} \left( \Pi_{i(1)}^{\gamma(l_1 \dots l_d)} \Phi_{\theta^\beta}^i{}_{l_1 X^1 \dots l_d X^d} \right)_{(1)} \equiv \omega^\beta \partial P_{(1)}^\gamma / \partial \theta^\beta$$

and disappear after integration w.r.t.  $\boldsymbol{\theta}$ . We can assume also, that the unessential phase shift  $\mathbf{S}(\mathbf{X})/\epsilon$  in the integrands above is omitted after taking all the differentiations w.r.t.  $\mathbf{X}$ .

From the other hand, we can write for the time derivative of the average  $\langle P^\gamma \rangle$ :

$$\langle P^\gamma \rangle_T \equiv \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{l_1, \dots, l_d} \Pi_{i(0)}^{\gamma(l_1 \dots l_d)} \left( k_1^{\gamma_1^1} \dots k_1^{\gamma_{l_1}^1} \dots k_d^{\gamma_1^d} \dots k_d^{\gamma_{l_d}^d} \Phi_{\theta^{\gamma_1^1} \dots \theta^{\gamma_{l_1}^1} \dots \theta^{\gamma_1^d} \dots \theta^{\gamma_{l_d}^d}}^i \right)_T \frac{d^m \theta}{(2\pi)^m}$$

Putting  $k_p^\alpha = S_{X^p}^\alpha$  and using relations (3.2), we can see then after the integration by parts that the relations

$$\langle P^\gamma \rangle_T - \langle Q^{\gamma^1} \rangle_{X^1} - \dots - \langle Q^{\gamma^d} \rangle_{X^d} = 0$$

can be actually written in the form:

$$\int_0^{2\pi} \dots \int_0^{2\pi} \zeta_{i[\mathbf{S}_\mathbf{x}, \mathbf{U}(\mathbf{x}, T)]}^{(\gamma)}(\boldsymbol{\theta}) \left( \Phi_T^i(\boldsymbol{\theta}, \mathbf{S}_\mathbf{x}, \mathbf{U}(\mathbf{x}, T)) - F_{[1]}^i(\boldsymbol{\theta}, \mathbf{x}, T) \right) \frac{d^m \theta}{(2\pi)^m} = 0$$

We can see now, that the expressions above represent the orthogonality of the covectors  $\zeta_{i[\mathbf{S}_\mathbf{x}, \mathbf{U}(\mathbf{x}, T)]}^{(\gamma)}(\boldsymbol{\theta})$  to the discrepancy  $\mathbf{f}_{(1)}(\boldsymbol{\theta}, \mathbf{x}, T)$ , introduced in (3.4). From Proposition 3.1 we get then the statement of the Lemma.

Lemma 3.1 is proved.

For our further considerations we will need also the definitions, characterizing the Hamiltonian properties of a complete regular family of  $m$ -phase solutions of (2.2).

**Definition 3.2.**

We call a complete regular family  $\Lambda$  a complete Hamiltonian family of  $m$ -phase solutions of (2.2) if it represents a regular Hamiltonian submanifold in the space of quasiperiodic functions according to Definition 2.1.

**Definition 3.3.**

We say that a complete Hamiltonian family  $\Lambda$  of  $m$ -phase solutions of system (2.2) is equipped with a minimal set of commuting integrals if it represents a regular Hamiltonian submanifold equipped with a minimal set of commuting integrals according to Definition 2.2.

We want to suggest now the Hamiltonian structure for the Whitham system (3.12) under the requirement that the family  $\Lambda$  represents a complete Hamiltonian family of  $m$ -phase solutions of system (2.2) equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . As we will see, the Hamiltonian structures for system (3.12) can be considered as a deformation of the structures, arising in Chapter 2, with the aid of the Dubrovin - Novikov procedure. Let us describe now the corresponding scheme.

As we said already, we assume that the integrals  $I^\gamma$  have the form (2.12) with some densities  $P^\gamma(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots)$ . In complete analogy with the Dubrovin - Novikov scheme, we can represent the pairwise Poisson brackets of the densities  $P^\gamma(\mathbf{x})$ ,  $P^\rho(\mathbf{y})$  in the form

$$\{P^\gamma(\mathbf{x}), P^\rho(\mathbf{y})\} = \sum_{l_1, \dots, l_d} A_{l_1 \dots l_d}^{\gamma\rho}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d)$$

$(l_1, \dots, l_d \geq 0)$ .

View relations (2.13) we can also write here

$$A_{0 \dots 0}^{\gamma\rho}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) \equiv \partial_{x^1} Q^{\gamma\rho^1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) + \dots + \partial_{x^d} Q^{\gamma\rho^d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_\mathbf{x}, \dots) \quad (3.15)$$

for some functions  $(Q^{\gamma\rho 1}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots), \dots, Q^{\gamma\rho d}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots))$ .

Let us define now the averaged Poisson bracket  $\{\dots, \dots\}_{\text{AV}}$  on the space of fields  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  by the formula

$$\begin{aligned} \{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\}_{\text{AV}} &= 0 \quad , \quad \alpha, \beta = 1, \dots, m, \\ \{S^\alpha(\mathbf{X}), U^\gamma(\mathbf{Y})\}_{\text{AV}} &= \omega^{\alpha\gamma}(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y}) \quad , \\ \{U^\gamma(\mathbf{X}), U^\rho(\mathbf{Y})\}_{\text{AV}} &= \langle A_{10\dots 0}^{\gamma\rho}(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})) \delta_{X^1}(\mathbf{X} - \mathbf{Y}) + \dots + \\ &+ \langle A_{0\dots 01}^{\gamma\rho}(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})) \delta_{X^d}(\mathbf{X} - \mathbf{Y}) + \\ &+ [\langle Q^{\gamma\rho p}(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})) \rangle]_{X^p} \delta(\mathbf{X} - \mathbf{Y}) \quad , \quad \gamma, \rho = 1, \dots, m+s \end{aligned} \quad (3.16)$$

In the next chapter we will prove the Jacobi identity for the bracket (3.16) and show that system (3.12) is Hamiltonian with respect to bracket (3.16) with the Hamiltonian functional

$$H_{av} = \int \langle P_H \rangle(\mathbf{S}_{\mathbf{x}}, U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) d^d X \quad (3.17)$$

In the main features, the next chapter will be of a technical nature.

Let us note now, that the construction of the Poisson bracket (3.16) can be also justified in terms of the Dubrovin - Novikov scheme in the presence of a “complete” set of commuting integrals  $(I^1, \dots, I^N)$ ,  $N = m(d+1)+s$  (see [41]). However, as we said already, the presence of a complete set of commuting integrals can be in fact too strict in the multi-dimensional situation. All the considerations here will involve just the presence of a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ , which is a much weaker requirement in general case.

## 4 The averaging of a local Poisson bracket.

In this chapter we will consider in detail the proof of the Jacobi identity for the averaged Poisson bracket. The proof will follow the general idea of the Dirac restriction of a Poisson bracket on a submanifold and have some common features with the considerations of Chapter 2. Let us note also that the considerations of this chapter will contain quite a lot of technical calculations.

First of all, we will introduce here a special submanifold  $\mathcal{K}$  in the space of functions  $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X})$  consisting of the functions

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) = \Phi^i\left(\boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X})\right) \quad ,$$

such that the functions  $\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$  represent the functions from the family  $\Lambda$  at every fixed  $\mathbf{X}$ . As we can see, the functions  $\mathbf{S}(\mathbf{X}) = (S^1(\mathbf{X}), \dots, S^m(\mathbf{X}))$  and  $\mathbf{U}(\mathbf{X}) = (U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$  represent a special “coordinate system” on the submanifold  $\mathcal{K}$ , depending on the parameter  $\epsilon$ .

Like in Chapter 2, we will introduce here special functionals  $(\mathbf{S}^{[\zeta]}(\mathbf{X}), \mathbf{U}^{[\zeta]}(\mathbf{X}))$  (for any fixed values  $\boldsymbol{\zeta}(\mathbf{X}) = (\zeta^1(\mathbf{X}), \dots, \zeta^m(\mathbf{X}))$  representing the coordinates  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  after the restriction on  $\mathcal{K}$  and defined near the “points”

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) = \Phi^i\left(\boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \boldsymbol{\zeta}_{X^1}, \dots, \boldsymbol{\zeta}_{X^d}, \mathbf{U}(\mathbf{X})\right) \in \mathcal{K}$$



in the functional space.

As will be shown below, for any regular Hamiltonian family  $\Lambda$ , equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ , the pairwise Poisson brackets of the functionals  $(\mathbf{S}^{[\zeta]}(\mathbf{X}), \mathbf{U}^{[\zeta]}(\mathbf{X}))$  on  $\mathcal{K}$  coincide in the main order of  $\epsilon$  with the relations (3.16). Besides that, it can be shown, that for the functionals  $(\mathbf{S}^{[\zeta]}(\mathbf{X}), \mathbf{U}^{[\zeta]}(\mathbf{X}))$  defined in the right way the higher terms in  $\epsilon$  of their pairwise Poisson brackets demonstrate necessary regularity properties in the vicinity of the submanifold  $\mathcal{K}$ . The proof of the corresponding statements is based on some technical calculations and is represented in the first part of the chapter.

The proof of the Jacobi identity for the bracket (3.16) is based on the idea of the Dirac restriction of a Poisson bracket on a submanifold and requires the resolvability of some special linear systems of PDE's on the submanifold  $\mathcal{K}$ . The corresponding systems are connected with the functionals  $(\mathbf{S}^{[\zeta]}(\mathbf{X}), \mathbf{U}^{[\zeta]}(\mathbf{X}))$  on the submanifold  $\mathcal{K}$  and represent systems of PDE's on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions  $\beta_i(\theta^1, \dots, \theta^m)$  at every fixed  $\mathbf{X}$ . The operators of the systems are given now by the main terms of the pairwise Poisson brackets of constraints  $g^i(\boldsymbol{\theta}, \mathbf{X})$  on the submanifold  $\mathcal{K}$  and we have actually here the situation similar to that considered in Chapter 2. It can be shown again, that under the assumptions of our scheme the right-hand parts of these systems are automatically orthogonal to all the “regular” (left) eigen-vectors of the system operators corresponding to the zero eigen-value, which provides in fact the resolvability of these systems in a large set of “regular” examples. In particular, we can claim the “regular” behavior of the solutions of these systems in the single-phase case and also for a special class of rather simple Poisson brackets (2.6).

However, in the general multi-phase case the situation is more complicated and the systems mentioned above can be actually unresolvable at some “points” of the submanifold  $\mathcal{K}$  because of rather irregular behavior of the spectra of the system operators. Fortunately, for the proof of the Jacobi identity for the bracket (3.16) it is enough to require again the solvability of the mentioned systems just on some dense set of parameters on the family  $\Lambda$ . Theorem 4.1 is devoted to the proof of the Jacobi identity for the bracket (3.16) under the corresponding assumptions. Let us say here also, that the assumptions of Theorem 4.1 seem in fact to be rather natural in the multi-phase situation and are satisfied in the known examples.

Theorem 4.2 is devoted to the proof of the invariance of the procedure of bracket averaging with respect to the choice of the set of functionals  $\{I^\gamma\}$ .

Finally, in Theorem 4.3 we prove that the averaged bracket (3.16) gives the Hamiltonian structure for the system (3.12) with the Hamiltonian functional (3.17).

Let us start now the detailed consideration of the stated scheme. First, we have to introduce the extended space of fields making the change

$$\varphi(\mathbf{x}) \rightarrow \varphi(\boldsymbol{\theta}, \mathbf{x})$$

where the new functions  $\varphi(\boldsymbol{\theta}, \mathbf{x})$  are  $2\pi$ -periodic with respect to each  $\theta^\alpha$  at every fixed values of  $\mathbf{X}$ .

Easy to see that we can define the Poisson bracket on the extended field space by expression

$$\{\varphi^i(\boldsymbol{\theta}, \mathbf{x}), \varphi^j(\boldsymbol{\theta}', \mathbf{y})\} = \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\varphi, \varphi_{\mathbf{x}}, \dots) \delta^{(l_1)}(x^1 - y^1) \dots \delta^{(l_d)}(x^d - y^d) \delta(\boldsymbol{\theta} - \boldsymbol{\theta}')$$

After the replacement  $\mathbf{x} \rightarrow \mathbf{X} = \epsilon \mathbf{x}$  we can then introduce the Poisson bracket

$$\begin{aligned} \{\varphi^i(\boldsymbol{\theta}, \mathbf{X}), \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} &= \\ &= \sum_{l_1, \dots, l_d} \epsilon^{l_1 + \dots + l_d} B_{(l_1, \dots, l_d)}^{ij}(\varphi, \epsilon \varphi_{\mathbf{x}}, \dots) \delta^{(l_1)}(X^1 - Y^1) \dots \delta^{(l_d)}(X^d - Y^d) \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \quad (4.1) \end{aligned}$$

on the space of fields  $\varphi(\boldsymbol{\theta}, \mathbf{X})$ .

Let us describe now the submanifold  $\mathcal{K}$  in the extended field space connected with the slow modulated  $m$ -phase solutions of system (2.2). Namely, we will assume that the functions  $\varphi(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$  represent functions from the family  $\hat{\Lambda}$  at any fixed value of  $\mathbf{X}$ . Besides that, we set, that the functions

$$\mathbf{S}(\mathbf{X}) = (S^1(\mathbf{X}), \dots, S^m(\mathbf{X})) \quad , \quad \mathbf{U}(\mathbf{X}) = (U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$$

give a “coordinate system” on the submanifold  $\mathcal{K}$  such that the functions  $\varphi(\boldsymbol{\theta}, \mathbf{X}) \in \mathcal{K}$  have the form

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) = \Phi^i \left( \frac{\mathbf{S}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}) \right) \quad (4.2)$$

We can see that, according to our definition, the parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d)$  on  $\hat{\Lambda}$  are directly connected here with the derivatives  $(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d})$  of the functions  $(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}))$ .

Like in Chapter 2, we will need to introduce here the functionals on the extended field space giving the values of the parameters  $\mathbf{S}(\mathbf{X})$  and  $\mathbf{U}(\mathbf{X})$  on the submanifold  $\mathcal{K}$ .

Let us define the following functionals on the space of functions  $\varphi(\boldsymbol{\theta}, \mathbf{X})$ :

$$J^\gamma(\mathbf{X}) = \int_0^{2\pi} \dots \int_0^{2\pi} P^\gamma(\varphi, \epsilon \varphi_{\mathbf{X}}, \epsilon^2 \varphi_{\mathbf{X}\mathbf{X}}, \dots) \frac{d^m \theta}{(2\pi)^m} \quad , \quad \gamma = 1, \dots, m+s \quad (4.3)$$

It's not difficult to see that we can write for the values of  $J^\gamma(\mathbf{X})$  on the submanifold  $\mathcal{K}$ :

$$J^\gamma(\mathbf{X}) = U^\gamma(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l J_{(l)}^\gamma(\mathbf{X}, [\mathbf{S}_{\mathbf{X}}, \mathbf{U}]) \quad , \quad \gamma = 1, \dots, m+s \quad (4.4)$$

where  $J_{(l)}^\gamma(\mathbf{X}, [\mathbf{S}_{\mathbf{X}}, \mathbf{U}])$  - are polynomials in the derivatives  $(\mathbf{U}_{\mathbf{X}}, \mathbf{U}_{\mathbf{X}\mathbf{X}}, \dots)$  and  $(\mathbf{S}_{\mathbf{X}\mathbf{X}}, \mathbf{S}_{\mathbf{X}\mathbf{X}\mathbf{X}}, \dots)$  with coefficients depending on the values  $\mathbf{U}(\mathbf{X})$  and  $\mathbf{k}_q(\mathbf{X}) = \mathbf{S}_{X^q}$  on  $\mathcal{K}$ . Easy to see also that the right-hand part of expression (4.4) does not contain an explicit dependence on the values  $\mathbf{S}(\mathbf{X})/\epsilon$  view the invariance of the functionals  $J^\gamma(\mathbf{X})$  w.r.t. the transformation

$$\mathbf{S}(\mathbf{X}) \rightarrow \mathbf{S}(\mathbf{X}) + \text{const}$$

Let us introduce here the following gradation rule. Namely, we prescribe gradation degree 0 to the functions  $\mathbf{U}(\mathbf{X})$  and  $\mathbf{k}_q(\mathbf{X}) = \mathbf{S}_{X^q}$  on  $\mathcal{K}$  while every differentiation w.r.t.  $\mathbf{X}$  increases the gradation degree by 1. The expression (4.4) represents then the graded expansion of the values of  $J^\gamma(\mathbf{X})$  on  $\mathcal{K}$ . Let us note also that the functions  $S^\alpha(\mathbf{X})$  have gradation degree  $-1$  in this scheme.

It is easy to see that the transformations (4.4) can be inverted as a formal series in  $\epsilon$  at every fixed  $\mathbf{S}(\mathbf{X})$ , such that we can write

$$U^\gamma(\mathbf{X}) = J^\gamma(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l U_{(l)}^\gamma(\mathbf{X}, [\mathbf{S}_{\mathbf{X}}, \mathbf{J}]) \quad , \quad \gamma = 1, \dots, m+s \quad (4.5)$$

on the functions of the submanifold  $\mathcal{K}$ .

The functions  $U_{(l)}^\gamma(\mathbf{X}, [\mathbf{S}_{\mathbf{X}}, \mathbf{J}])$  are given here by polynomials in  $(\mathbf{J}_{\mathbf{X}}, \mathbf{J}_{\mathbf{X}\mathbf{X}}, \dots)$  and  $(\mathbf{S}_{\mathbf{X}\mathbf{X}}, \mathbf{S}_{\mathbf{X}\mathbf{X}\mathbf{X}}, \dots)$  with coefficients depending on the values of  $\mathbf{J}$  and  $\mathbf{S}_{\mathbf{X}}$  at the point  $\mathbf{X}$ . Now the functions  $U_{(l)}^\gamma$  have degree  $l$  according to analogous rule applied to the functions  $\mathbf{J}(\mathbf{X})$  and  $\mathbf{S}_{\mathbf{X}}$ . Let us note that this gradation rule does not coincide exactly with the rule formulated above view the nontrivial connection between the values of  $\mathbf{U}(\mathbf{X})$  and  $\mathbf{J}(\mathbf{X})$  on  $\mathcal{K}$ .

Let us now fix the values  $S^\alpha(\mathbf{X}) = \zeta^\alpha(\mathbf{X})$  on the submanifold  $\mathcal{K}$ . Like in Chapter 2, we are going to define the local functionals  $S^{\alpha[\zeta]}(\mathbf{X})$  in the region of the extended field space near the point  $(\mathbf{U}(\mathbf{X}), \zeta(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Let us introduce here the functionals

$$\vartheta_\alpha^{[\zeta]}(\mathbf{X}) = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \varphi^i(\boldsymbol{\theta}, \mathbf{X}) \Phi_{\theta^\alpha}^i \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{X^1}, \dots, \zeta_{X^d}, \mathbf{J}(\mathbf{X}) \right) \frac{d^m \theta}{(2\pi)^m}$$

For the functions

$$\varphi^i(\boldsymbol{\theta}, \mathbf{X}) = \Phi^i \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}) \right),$$

where  $\mathbf{S}(\mathbf{X}) = \zeta(\mathbf{X}) + \epsilon \Delta \boldsymbol{\theta}_0(\mathbf{X})$ , we then have

$$\vartheta_\alpha^{[\zeta]}(\mathbf{X}) = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi^i(\boldsymbol{\theta} + \Delta \boldsymbol{\theta}_0(\mathbf{X}), \zeta_{\mathbf{X}} + \epsilon \Delta \boldsymbol{\theta}_{0\mathbf{X}}, \mathbf{U}(\mathbf{X})) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \zeta_{\mathbf{X}}, \mathbf{J}(\mathbf{X})|_{\mathcal{K}}) \frac{d^m \theta}{(2\pi)^m}$$

View relations (4.5) and (2.19) on  $\mathcal{K}$  we can see that the mapping

$$\Delta \boldsymbol{\theta}_0(\mathbf{X}) \rightarrow \boldsymbol{\vartheta}^{[\zeta]}(\mathbf{X})|_{\mathcal{K}}$$

is locally invertible in the main order of  $\epsilon$  at any fixed values of  $\mathbf{J}(\mathbf{X})$ . As a result, we can locally represent the values of  $\Delta \boldsymbol{\theta}_0(\mathbf{X})$  on  $\mathcal{K}$  in the form of the asymptotic expansions:

$$\begin{aligned} \Delta \boldsymbol{\theta}_0(\mathbf{X}) &= \tau^\alpha \left( \vartheta_1^{[\zeta]}(\mathbf{X})|_{\mathcal{K}}, \dots, \vartheta_m^{[\zeta]}(\mathbf{X})|_{\mathcal{K}}, J^1(\mathbf{X})|_{\mathcal{K}}, \dots, J^{m+s}(\mathbf{X})|_{\mathcal{K}} \right) + \\ &+ \sum_{l \geq 1} \epsilon^l \Delta \theta_{0(l)}^\alpha(\mathbf{X}, [\boldsymbol{\vartheta}^{[\zeta]}(\mathbf{X})|_{\mathcal{K}}, \mathbf{J}(\mathbf{X})|_{\mathcal{K}}, \zeta_{\mathbf{X}}]) \end{aligned}$$

where  $\Delta \theta_{0(l)}^\alpha(\mathbf{X})$  are some local functions of their arguments and their derivatives, polynomial in the derivatives and having gradation degree  $l$  in terms of the total number of differentiations of the functions  $\boldsymbol{\vartheta}^{[\zeta]}(\mathbf{X})|_{\mathcal{K}}$ ,  $\mathbf{J}(\mathbf{X})|_{\mathcal{K}}$ , and  $\zeta_{X^q}$  w.r.t.  $\mathbf{X}$ .

Let us define now the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$  by the asymptotic series

$$\begin{aligned} \tilde{S}^{\alpha[\zeta]}(\mathbf{X}) &= \zeta(\mathbf{X}) + \epsilon \tau^\alpha \left( \vartheta_1^{[\zeta]}(\mathbf{X}), \dots, \vartheta_m^{[\zeta]}(\mathbf{X}), J^1(\mathbf{X}), \dots, J^{m+s}(\mathbf{X}) \right) + \\ &+ \epsilon \sum_{l \geq 1} \epsilon^l \Delta \theta_{0(l)}^\alpha(\mathbf{X}, [\boldsymbol{\vartheta}^{[\zeta]}(\mathbf{X}), \mathbf{J}(\mathbf{X}), \zeta_{\mathbf{X}}]) \end{aligned} \quad (4.6)$$

Substituting the functionals  $\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{X})$  in (4.5) we can then also define the functionals

$$\tilde{U}^{\gamma[\zeta]}(\mathbf{X}) = J^\gamma(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l U_{(l)}^\gamma(\mathbf{X}, [\tilde{\mathbf{S}}_{\mathbf{X}}^{[\zeta]}, \mathbf{J}]) \quad , \quad \gamma = 1, \dots, m+s \quad (4.7)$$

in the neighborhood of the “points”  $(\mathbf{U}(\mathbf{X}), \zeta(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Like in Chapter 2, we can introduce here also the constraints

$$\tilde{g}^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) = \varphi^i(\boldsymbol{\theta}, \mathbf{X}) - \Phi^i \left( \boldsymbol{\theta} + \frac{\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{X})}{\epsilon}, \tilde{\mathbf{S}}_{X^1}^{[\zeta]}, \dots, \tilde{\mathbf{S}}_{X^d}^{[\zeta]}, \tilde{\mathbf{U}}^{[\zeta]}(\mathbf{X}) \right)$$

The constraints  $\tilde{g}^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$  are “numerated” now by the index  $i = 1, \dots, n$  and the “continuous indices”  $\theta^\alpha \in [0, 2\pi)$ ,  $X^q \in (-\infty, +\infty)$ .

Let us note here that we will mainly consider the “regularized” functionals

$$J_{[\mathbf{a}]} = \int J^\gamma(\mathbf{X}) a_\gamma(\mathbf{X}) d^d X \quad , \quad \tilde{U}_{[\mathbf{a}]}^{[\zeta]} = \int \tilde{U}^{\gamma[\zeta]}(\mathbf{X}) a_\gamma(\mathbf{X}) d^d X \quad ,$$

$$\tilde{S}_{[\mathbf{f}]}^{[\zeta]} = \int \tilde{S}^{\alpha[\zeta]}(\mathbf{X}) f_\alpha(\mathbf{X}) d^d X$$

with smooth compactly supported functions  $\mathbf{a}(\mathbf{X})$ ,  $\mathbf{f}(\mathbf{X})$ .

We can see that the “gradients” of the functionals  $\tilde{S}_{[\mathbf{f}]}^{[\zeta]}$  have the order  $O(\epsilon)$  on  $\mathcal{K}$  according to the definition of the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$ .

Easy to see that the functionals  $J_{[\mathbf{a}]}$  are invariant w.r.t. the transformations

$$\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) \rightarrow \boldsymbol{\varphi}(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \mathbf{X}) \quad (4.8)$$

As a corollary, we can write for the “gradients” of  $J_{[\mathbf{a}]}$ :

$$\mathcal{L}_\alpha \left[ \frac{\delta J_{[\mathbf{a}]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right] \equiv 0 \quad , \quad \alpha = 1, \dots, m$$

where  $\mathcal{L}_\alpha$  denote the Lie derivatives along the vector fields

$$\boldsymbol{\nu}_\alpha = \int \frac{d^m \theta}{(2\pi)^m} d^d X \quad \varphi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{X}) \frac{\delta}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})}$$

corresponding to transformations (4.8).

Let us note that the functionals  $\tilde{U}_{[\mathbf{a}]}^{[\zeta]}$  do not posses in general the analogous property being defined with the aid of the functionals  $\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{X})$ .

Now, let us redefine the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$  putting

$$S^{\alpha[\zeta]}(\mathbf{X}) = \tilde{S}^{\alpha[\zeta]}(\mathbf{X}) +$$

$$+ \epsilon \int \tilde{h}_j^{\alpha[\zeta]} \left( \mathbf{X}, \boldsymbol{\theta} + \frac{\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{Y})}{\epsilon} - \frac{\boldsymbol{\zeta}(\mathbf{Y})}{\epsilon}, \mathbf{Y}; [\tilde{\mathbf{U}}^{[\zeta]}, \boldsymbol{\zeta}, \epsilon] \right) \tilde{g}^{j[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}) \frac{d^m \theta}{(2\pi)^m} d^d Y$$

where the distribution  $\tilde{h}_j^{\alpha[\zeta]}(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \mathbf{S}, \epsilon])$  is defined by the formula

$$\epsilon \tilde{h}_j^{\alpha[\zeta]}(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \mathbf{S}, \epsilon]) \equiv \left. \frac{\delta \tilde{S}^{\alpha[\zeta]}(\mathbf{X})}{\delta \varphi^j(\boldsymbol{\theta}, \mathbf{Y})} \right|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \mathbf{S}]}}$$

The last notation means here that the variation derivative of  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$  is taken on the function

$$\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{Y}) = \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon}, \mathbf{S}_{Y^1}, \dots, \mathbf{S}_{Y^d}, \mathbf{U}(\mathbf{Y}) \right)$$

from the submanifold  $\mathcal{K}$ .

According to their definition, the functionals  $S^{\alpha[\zeta]}(\mathbf{X})$  take the same values on the submanifold  $\mathcal{K}$  as the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$ .

Let us use now the smooth functions

$$\tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \mathbf{S}, \epsilon]) \equiv \int \tilde{h}_j^{\alpha[\zeta]}(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \mathbf{S}, \epsilon]) f_\alpha(\mathbf{X}) d^d X$$

with compactly supported  $f_\alpha(\mathbf{X})$ , corresponding to the regularized functionals  $\tilde{S}_{[\mathbf{f}]}^{[\zeta]}$ . From the definition of the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$  it's not difficult to get the following relations for  $\mathbf{S}(\mathbf{X}) \equiv \zeta(\mathbf{X})$ :

$$\tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) = \sum_{l \geq 0} \epsilon^l \tilde{h}_{j[l][\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon] \right)$$

where all  $\tilde{h}_{j[l][\mathbf{f}]}^{[\zeta]}$  are local functions of  $(\mathbf{U}(\mathbf{Y}), \zeta_{\mathbf{Y}})$  and their derivatives, polynomial in the derivatives  $(\mathbf{U}_{\mathbf{Y}}, \zeta_{\mathbf{Y}\mathbf{Y}}, \mathbf{U}_{\mathbf{Y}\mathbf{Y}}, \zeta_{\mathbf{Y}\mathbf{Y}\mathbf{Y}})$  and having degree  $l$  according to our gradation rule.

We can write on  $\mathcal{K}$ :

$$\begin{aligned} \left. \frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} &= \left. \frac{\delta \tilde{S}_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} + \epsilon \tilde{h}_{i[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon} - \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}; [\mathbf{U}, \zeta, \epsilon] \right) - \\ &- \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon} - \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon] \right) \Phi_{\theta^\alpha}^j \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon}, \mathbf{S}_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \times \\ &\quad \times \frac{1}{\epsilon} \left. \frac{\delta \tilde{S}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y - \\ &- \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon} - \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon] \right) \Phi_{k_q^\alpha}^j \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon}, \mathbf{S}_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \times \\ &\quad \times \left. \frac{\delta \tilde{S}_{Y^q}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y - \\ &- \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon} - \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon] \right) \Phi_{U^\gamma}^j \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{Y})}{\epsilon}, \mathbf{S}_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \times \\ &\quad \times \left. \frac{\delta \tilde{U}^{\gamma[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y \end{aligned}$$

In the case when the functions  $\zeta(\mathbf{X})$  and  $\mathbf{S}(\mathbf{X})$  belong to the same orbit of the group (4.8) (i.e.  $\zeta_{X^q} \equiv \mathbf{S}_{X^q}$ ,  $q = 1, \dots, d$ ), we can write

$$\left. \frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} = \left. \frac{\delta \tilde{S}_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} + \epsilon \tilde{h}_{i[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon} - \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}; [\mathbf{U}, \zeta, \epsilon] \right) -$$

$$\begin{aligned}
& - \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{\theta^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \times \\
& \quad \times \frac{1}{\epsilon} \frac{\delta \tilde{S}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y - \\
& - \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{k_q^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \times \\
& \quad \times \frac{\delta \tilde{S}_{Y^q}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y - \\
& - \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{U^\gamma}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \frac{\delta \tilde{U}^{\gamma[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y
\end{aligned}$$

Using the convolution with arbitrary variation  $\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  we can see then that the last term in the above expression is identically equal to zero, while the first term is canceled with the third and the forth terms according to the definition of the functions  $\tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon])$ . So, we can write for  $\mathbf{S}_{X^q} \equiv \zeta_{X^q}$ :

$$\frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} = \tilde{h}_{i[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon} - \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}; [\mathbf{U}, \zeta, \epsilon] \right)$$

in the full analogy with Chapter 2.

Easy to see that the 1-form  $\delta S_{[\mathbf{f}]}^{[\zeta]}/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  is invariant in this case with respect to the transformations (4.8), so we can write

$$\mathcal{L}_\alpha \left[ \frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right] \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} \equiv 0 \quad (4.10)$$

In general, we can write:

$$\frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} = \tilde{h}_{i[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon} - \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}; [\mathbf{U}, \zeta, \epsilon] \right) + \quad (4.11)$$

$$\begin{aligned}
& + \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \times \\
& \quad \times \left( \Phi_{\theta^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{S}_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) - \Phi_{\theta^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_{\mathbf{Y}}, \mathbf{U}(\mathbf{Y}) \right) \right) \times \\
& \quad \times \frac{1}{\epsilon} \frac{\delta \tilde{S}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y +
\end{aligned}$$

$$\begin{aligned}
& + \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \times \\
& \quad \times \left( \Phi_{k_q^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{S}_Y, \mathbf{U}(\mathbf{Y}) \right) - \Phi_{k_q^\alpha}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_Y, \mathbf{U}(\mathbf{Y}) \right) \right) \times \\
& \quad \times \left. \frac{\delta \tilde{S}_{Y^q}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y + \\
& + \epsilon \int \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \times \\
& \quad \times \left( \Phi_{U^\gamma}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \mathbf{S}_Y, \mathbf{U}(\mathbf{Y}) \right) - \Phi_{U^\gamma}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_Y, \mathbf{U}(\mathbf{Y}) \right) \right) \times \\
& \quad \times \left. \frac{\delta \tilde{U}^{\gamma[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y
\end{aligned}$$

or, for close values of  $\zeta_Y$  and  $\mathbf{S}_Y$ :

$$\left. \frac{\delta S_{[\mathbf{f}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} = \tilde{h}_{i[\mathbf{f}]}^{[\zeta]} \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon} - \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}; [\mathbf{U}, \zeta, \epsilon] \right) + \quad (4.12)$$

$$\begin{aligned}
& + \epsilon \int \left( S_{Y^p}^\beta - \zeta_{Y^p}^\beta \right) \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{\theta^\alpha k_p^\beta}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_Y, \mathbf{U}(\mathbf{Y}) \right) \times \\
& \quad \times \frac{1}{\epsilon} \left. \frac{\delta \tilde{S}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y + \\
& + \epsilon \int \left( S_{Y^p}^\beta - \zeta_{Y^p}^\beta \right) \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{k_q^\alpha k_p^\beta}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_Y, \mathbf{U}(\mathbf{Y}) \right) \times \\
& \quad \times \left. \frac{\delta \tilde{S}_{Y^q}^{\alpha[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y + \\
& + \epsilon \int \left( S_{Y^p}^\beta - \zeta_{Y^p}^\beta \right) \tilde{h}_{j[\mathbf{f}]}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{Y}; [\mathbf{U}, \zeta, \epsilon]) \Phi_{U^\gamma k_p^\beta}^j \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Y})}{\epsilon}, \zeta_Y, \mathbf{U}(\mathbf{Y}) \right) \times \\
& \quad \times \left. \frac{\delta \tilde{U}^{\gamma[\zeta]}(\mathbf{Y})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \mathbf{S}]}} \frac{d^m \theta}{(2\pi)^m} d^d Y + \\
& \quad + O(|\mathbf{S}_Y - \zeta_Y|^2)
\end{aligned}$$

Using the functionals  $S^{\alpha[\zeta]}(\mathbf{X})$  we can introduce also the new functionals

$$U^{\gamma[\zeta]}(\mathbf{X}) = J^\gamma(\mathbf{X}) + \sum_{l \geq 1} \epsilon^l U_{(l)}^\gamma \left( \mathbf{X}, [\mathbf{S}_X^{[\zeta]}, \mathbf{J}] \right) \quad , \quad \gamma = 1, \dots, m+s \quad (4.13)$$

and the new constraints

$$g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) = \varphi^i(\boldsymbol{\theta}, \mathbf{X}) - \Phi^i\left(\boldsymbol{\theta} + \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon}, \mathbf{S}_{X^1}^{[\zeta]}, \dots, \mathbf{S}_{X^d}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X})\right) \quad (4.14)$$

The equations

$$g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) = 0$$

define the submanifold  $\mathcal{K}$  near the points  $(\mathbf{U}(\mathbf{X}), \boldsymbol{\zeta}(\mathbf{X}))$ .

Like in Chapter 2, the constraints  $g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$  are also not independent here. Thus, in the same way, we can write here the following set of relations

$$\int \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta S^{\alpha[\zeta]}(\mathbf{Z})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\mathcal{K}} \frac{\delta g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})}{\delta \varphi^j(\boldsymbol{\theta}', \mathbf{Y})} \Big|_{\mathcal{K}} \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0 \quad , \quad \alpha = 1, \dots, m \quad (4.15)$$

$$\int \int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta U^{\gamma[\zeta]}(\mathbf{Z})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\mathcal{K}} \frac{\delta g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})}{\delta \varphi^j(\boldsymbol{\theta}', \mathbf{Y})} \Big|_{\mathcal{K}} \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0 \quad , \quad \gamma = 1, \dots, m + s \quad (4.16)$$

which take place identically for their “gradients” on the submanifold  $\mathcal{K}$ .

As we can see, the functionals  $\mathbf{U}^{[\zeta]}(\mathbf{X})$ ,  $\mathbf{S}^{[\zeta]}(\mathbf{X})$ , and  $\mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$  are introduced just as formal series in  $\epsilon$  near the points of the submanifold  $\mathcal{K}$  with the set of coordinates  $(\mathbf{U}(\mathbf{Z}), \boldsymbol{\zeta}(\mathbf{Z}))$ . We can see also, that the values of  $S^{\alpha[\zeta]}(\mathbf{X})$  are restricted by the conditions  $S^{\alpha}(\mathbf{X}) = \zeta^{\alpha}(\mathbf{X}) + O(\epsilon)$ , while their gradients have the order  $O(\epsilon)$  on  $\mathcal{K}$ . This definition will be sufficient for us here, since we are going to consider actually just the pairwise Poisson brackets of these functionals and their gradients at the points of the submanifold  $\mathcal{K}$  with coordinates  $(\mathbf{U}(\mathbf{Z}), \boldsymbol{\zeta}(\mathbf{Z}))$ .

The functionals  $(\mathbf{U}^{[\zeta]}(\mathbf{X}), \mathbf{S}^{[\zeta]}(\mathbf{X}), \mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X}))$  can be considered as an “overdetermined” coordinate system near the points  $(\mathbf{U}(\mathbf{Z}), \boldsymbol{\zeta}(\mathbf{Z}))$  of the submanifold  $\mathcal{K}$ , since we always have relations (4.15) - (4.16) for the gradients of the coordinates  $\mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$ .

Let us define regularized functionals  $g_{[\mathbf{Q}]}^{[\zeta]}$  by the formula

$$g_{[\mathbf{Q}]}^{[\zeta]} = \int \int_0^{2\pi} \dots \int_0^{2\pi} g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) Q_i\left(\frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X}\right) \frac{d^m \theta}{(2\pi)^m} d^d X$$

with smooth, compactly supported in  $\mathbf{X}$  and  $2\pi$ -periodic in each  $\theta^{\alpha}$  functions  $Q_i(\boldsymbol{\theta}, \mathbf{X})$ . In fact, it will be convenient to put some additional requirements on the functions  $Q_i(\boldsymbol{\theta}, \mathbf{X})$ . Namely, let us define for arbitrary smooth, compactly supported in  $\mathbf{X}$  and  $2\pi$ -periodic in each  $\theta^{\alpha}$  functions  $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X}) = (\tilde{Q}_1(\boldsymbol{\theta}, \mathbf{X}), \dots, \tilde{Q}_n(\boldsymbol{\theta}, \mathbf{X}))$  the functionals

$$\begin{aligned} Q_i(\boldsymbol{\theta}, \mathbf{X}) &= \tilde{Q}_i(\boldsymbol{\theta}, \mathbf{X}) - \Phi_{\theta^{\beta}}^i(\boldsymbol{\theta}, \mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X})) M^{\beta\gamma}(\mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X})) \times \\ &\quad \times \int_0^{2\pi} \dots \int_0^{2\pi} \tilde{Q}_j(\boldsymbol{\theta}', \mathbf{X}) \Phi_{\theta'^{\gamma}}^j(\boldsymbol{\theta}', \mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X})) \frac{d^m \theta'}{(2\pi)^m} \end{aligned}$$

Here the matrix  $M^{\beta\gamma}(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}))$  represents the inverse of the matrix

$$M_{\beta\gamma}(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) = \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi_{\theta^{\beta}}^i(\boldsymbol{\theta}, \mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \Phi_{\theta^{\gamma}}^i(\boldsymbol{\theta}, \mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \frac{d^m \theta}{(2\pi)^m}$$

on a complete regular family  $\Lambda$ .



By definition, the functionals  $Q_i(\boldsymbol{\theta}, \mathbf{X})$  represent local functions of  $(\mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X}))$

$$Q_i(\boldsymbol{\theta}, \mathbf{X}) \equiv Q_i(\boldsymbol{\theta}, \mathbf{X}, \mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X}))$$

and the arbitrary fixed functions  $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$ . We will assume everywhere below that  $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$  represents a functional of this type.

We can see that for fixed values of the functionals  $(\mathbf{S}^{[\zeta]}(\mathbf{Z}), \mathbf{U}^{[\zeta]}(\mathbf{Z}))$  the values of  $Q_i(\boldsymbol{\theta}, \mathbf{X})$  with arbitrary  $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$  represent all possible smooth compactly supported in  $\mathbf{X}$  and  $2\pi$ -periodic in each  $\theta^\alpha$  functions satisfying the restrictions

$$\int_0^{2\pi} \dots \int_0^{2\pi} Q_i(\boldsymbol{\theta}, \mathbf{X}) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{S}_{\mathbf{X}}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X})) \frac{d^m \theta}{(2\pi)^m} = 0 \quad , \quad \forall \mathbf{X}, \quad \alpha = 1, \dots, m \quad (4.17)$$

It's not difficult to see also that for the functionals (4.13) - (4.14), defined with the aid of  $\mathbf{S}^{[\zeta]}(\mathbf{X})$  and  $\mathbf{J}(\mathbf{X})$ , we can write the relations, analogous to (4.10), i.e.

$$\mathcal{L}_\alpha \left[ \frac{\delta U_{[\mathbf{q}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right] \Big|_{\varphi = \Phi_{[\mathbf{U}, \zeta]}} \equiv 0 \quad , \quad \mathcal{L}_\alpha \left[ \frac{\delta g_{[\mathbf{Q}]}^{[\zeta]}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right] \Big|_{\varphi = \Phi_{[\mathbf{U}, \zeta]}} \equiv 0 \quad (4.18)$$

Using the evident relation

$$\delta \mathcal{F} = \int \frac{\delta \mathcal{F}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \delta \varphi^i(\boldsymbol{\theta}, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} d^d X \quad (4.19)$$

and the specific definition of the constraints  $g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$ , it is easy to obtain the following relation at the points of the submanifold  $\mathcal{K}$ :

$$\begin{aligned} \delta \mathcal{F} = & \int \frac{\delta \mathcal{F}}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\mathcal{K}} \delta g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} d^d X + \\ & + \int [\delta \mathcal{F}|_{\mathcal{K}} / \delta U^\gamma(\mathbf{X})] \delta U^{\gamma[\zeta]}(\mathbf{X}) d^d X + \int [\delta \mathcal{F}|_{\mathcal{K}} / \delta S^\alpha(\mathbf{X})] \delta S^{\alpha[\zeta]}(\mathbf{X}) d^d X \end{aligned} \quad (4.20)$$

for any functional  $\mathcal{F}$ . Let us note that here and below we always assume that the derivatives  $\delta / \delta U^\gamma(\mathbf{X})$  and  $\delta / \delta S^\alpha(\mathbf{X})$  are taken under the conditions  $\mathbf{g}(\boldsymbol{\theta}, \mathbf{W}) = 0$  (all  $\boldsymbol{\theta}$  and  $\mathbf{W}$ ).

We need to discuss now the pairwise Poisson brackets of the functionals  $\mathbf{J}(\mathbf{X})$ ,  $\mathbf{U}^{[\zeta]}(\mathbf{X})$ ,  $\mathbf{S}^{[\zeta]}(\mathbf{X})$ , and  $\mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$  on the submanifold  $\mathcal{K}$ .

Let us note that, according to (3.15), we can write here

$$A_{0\dots 0}^{\gamma\rho}(\varphi, \epsilon\varphi_{\mathbf{X}}, \dots) \equiv \epsilon \partial_{X^1} Q^{\gamma\rho 1}(\varphi, \epsilon\varphi_{\mathbf{X}}, \dots) + \dots + \epsilon \partial_{X^d} Q^{\gamma\rho d}(\varphi, \epsilon\varphi_{\mathbf{X}}, \dots) \quad (4.21)$$

Using the definition of the functionals  $J^\gamma(\mathbf{X})$  and relations (4.21) we can then write for the functionals

$$J_{[\mathbf{a}]} = \int J^\gamma(\mathbf{X}) a_\gamma(\mathbf{X}) d^d X \quad , \quad J_{[\mathbf{b}]} = \int J^\gamma(\mathbf{X}) b_\gamma(\mathbf{X}) d^d X$$

the relations

$$\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}|_{\mathcal{K}} = \epsilon \{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}|_{\mathcal{K}[1]} + \epsilon^2 \{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}|_{\mathcal{K}[2]} + \dots \quad (4.22)$$

The functionals  $\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}[l]}$  depend on the coordinates  $[\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X})]$  on  $\mathcal{K}$  and are invariant under the shift

$$S^\alpha(\mathbf{X}) \rightarrow S^\alpha(\mathbf{X}) + \text{const} \quad (4.23)$$

Let us say also, that all the statements above are also valid for the functions  $\{J^\gamma(\mathbf{X}), J_{[\mathbf{b}]}\}_{|\mathcal{K}[l]}$  and the distributions  $\{J^\gamma(\mathbf{X}), J^\rho(\mathbf{Y})\}_{|\mathcal{K}[l]}$  on  $\mathcal{K}$ .

Using the expressions

$$\begin{aligned} \left. \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}} &\equiv \\ &\equiv \sum_{l_1, \dots, l_d} \int \left[ \frac{\delta}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{W})} \int_0^{2\pi} \dots \int_0^{2\pi} \epsilon^{l_1 + \dots + l_d} A_{l_1 \dots l_d}^{\gamma\rho}(\boldsymbol{\varphi}(\boldsymbol{\theta}', \mathbf{X}), \dots) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \right]_{\mathcal{K}} \times \\ &\times a_\gamma(\mathbf{X}) b_{\rho, l_1 X^1 \dots l_d X^d}(\mathbf{X}) d^d X \quad (4.24) \end{aligned}$$

we can represent the values (4.24) as the graded decompositions on  $\mathcal{K}$  depending on the values  $k_q^\alpha(\mathbf{X}) = S_{X^q}^\alpha, U^\gamma(\mathbf{X})$  and their derivatives at the corresponding point  $\mathbf{W}$ .

According to (4.21), we can claim again that the corresponding expansion of (4.24) starts with the first degree of  $\epsilon$ :

$$\left. \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}} = \epsilon \left. \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[1]} + \epsilon^2 \left. \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{W})} \right|_{\mathcal{K}[2]} + \dots \quad (4.25)$$

Every term in decomposition (4.25) represents a local function of  $k_q^\alpha(\mathbf{W}) = S_{W^q}^\alpha, U^\gamma(\mathbf{W}), a_\gamma(\mathbf{W}), b_\gamma(\mathbf{W})$  and their derivatives w.r.t.  $\mathbf{W}$ . All the terms in (4.25) are polynomial in the derivatives of  $(\mathbf{k}_q(\mathbf{W}), \mathbf{U}(\mathbf{W}), \mathbf{a}(\mathbf{W}), \mathbf{b}(\mathbf{W}))$  and have degree  $l$  given by the total number of differentiations of these functions w.r.t.  $\mathbf{W}$ . At the same time the dependence of the terms of (4.25) on the variables  $\boldsymbol{\theta}$  appears with the common shift  $\mathbf{S}(\mathbf{W})/\epsilon$ .

Easy to see that we have the relations

$$\frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}}}{\delta S^\alpha(\mathbf{W})} = O(\epsilon) \quad , \quad \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}}}{\delta U^\gamma(\mathbf{W})} = O(\epsilon) \quad (4.26)$$

Besides that, we can write

$$\frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}}}{\delta S^\alpha(\mathbf{W})}^{[1]} = \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}[1]}}{\delta S^\alpha(\mathbf{W})} \quad , \quad \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}}}{\delta U^\gamma(\mathbf{W})}^{[1]} = \frac{\delta\{J_{[\mathbf{a}]}, J_{[\mathbf{b}]}\}_{|\mathcal{K}[1]}}{\delta U^\gamma(\mathbf{W})} \quad (4.27)$$

on the submanifold  $\mathcal{K}$ .

In particular, all the relations (4.25) - (4.27) can be used at the “points”

$$\boldsymbol{\varphi}(\boldsymbol{\theta}, \mathbf{X}) = \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \boldsymbol{\zeta}_\mathbf{X}, \mathbf{U}(\mathbf{X}) \right)$$

of the submanifold  $\mathcal{K}$ .

From the definition of the functionals  $\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{X})$  and  $\mathbf{S}^{[\zeta]}(\mathbf{X})$  it's not difficult to get the following relations

$$\left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} = \epsilon \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}[0]} + \epsilon^2 \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}[1]} + \dots \quad (4.28)$$

$$\left\{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = \epsilon^2 \left\{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}^{[0]}} + \dots \quad (4.29)$$

where all the terms represent local graded expressions of  $\zeta_{\mathbf{X}}$ ,  $\mathbf{U}(\mathbf{X})$ ,  $\mathbf{f}(\mathbf{X})$ ,  $\mathbf{h}(\mathbf{X})$ ,  $\mathbf{b}(\mathbf{X})$  and their derivatives integrated over  $\mathbb{R}^d$ . In the same way, it is not difficult to get also at  $\mathbf{S}(\mathbf{X}) \equiv \zeta(\mathbf{X})$ :

$$\frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = \epsilon \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}^{[0]}} + \epsilon^2 \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}^{[1]}} + \dots \quad (4.30)$$

$$\frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = \epsilon^2 \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{W})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}^{[0]}} + \dots \quad (4.31)$$

Every term in (4.30) - (4.31) represents a local function of  $k_q^\alpha(\mathbf{W}) = \zeta_{W^q}^\alpha$ ,  $U^\gamma(\mathbf{W})$ ,  $b_\gamma(\mathbf{W})$ ,  $f_\alpha(\mathbf{W})$ ,  $h_\alpha(\mathbf{W})$  and their derivatives w.r.t.  $\mathbf{W}$ . All the terms again are polynomial in the derivatives of  $(\mathbf{k}_q(\mathbf{W}), \mathbf{U}(\mathbf{W}), \mathbf{b}(\mathbf{W}), \mathbf{f}(\mathbf{W}), \mathbf{h}(\mathbf{W}))$  with the corresponding gradation degree. The dependence of (4.30) and (4.31) on the variables  $\boldsymbol{\theta}$  appears with the common shift  $\zeta(\mathbf{W})/\epsilon$ .

Let us say now that the Poisson brackets containing the functionals  $\tilde{\mathbf{S}}^{[\zeta]}(\mathbf{X})$  and  $\mathbf{S}^{[\zeta]}(\mathbf{X})$  have in fact a little bit more complicated structure at the points of  $\mathcal{K}$  for general values of the coordinates  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$ . This circumstance is caused by the fact that, according to the definition, the densities of the functionals  $\vartheta^{[\zeta]}(\mathbf{X})$  contain an explicit dependence on the variables  $\boldsymbol{\theta}$ . Calculating the pairwise brackets of  $\vartheta_\alpha^{[\zeta]}(\mathbf{X})$  with  $J_{[\mathbf{b}]}$  and  $\vartheta_\beta^{[\zeta]}(\mathbf{Y})$  everywhere on  $\mathcal{K}$  we can represent them in the form:

$$\epsilon \left\{ \vartheta_\alpha^{[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}} = \epsilon \left\{ \vartheta_\alpha^{[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[0]}} + \epsilon^2 \left\{ \vartheta_\alpha^{[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[1]}} + \dots \quad (4.32)$$

$$\epsilon^2 \left\{ \vartheta_\alpha^{[\zeta]}(\mathbf{X}), \vartheta_\beta^{[\zeta]}(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} = \epsilon^2 \left\{ \vartheta_\alpha^{[\zeta]}(\mathbf{X}), \vartheta_\beta^{[\zeta]}(\mathbf{Y}) \right\} \Big|_{\mathcal{K}^{[0]}} + \dots \quad (4.33)$$

The terms in (4.32) - (4.33) have the decreasing orders in  $\epsilon$  like in (4.22), with the analogous dependence on the derivatives of the functions  $\mathbf{k}_q(\mathbf{X}) = \mathbf{S}_{X^q}$ ,  $\mathbf{U}(\mathbf{X})$  and  $\zeta_{\mathbf{X}}$ . However, the dependence on the values  $\mathbf{S}(\mathbf{X})$  on  $\mathcal{K}$  here is more complicated. Namely, in addition to the dependence on the values  $(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}), \zeta_{\mathbf{X}})$  and their derivatives, the terms in (4.32) - (4.33) can contain now also an explicit dependence on  $\Delta\boldsymbol{\theta}_0(\mathbf{X}) = (\mathbf{S}(\mathbf{X}) - \zeta(\mathbf{X}))/\epsilon$  due to the definition of the functionals  $\vartheta_\alpha^{[\zeta]}(\mathbf{X})$ . The corresponding dependence has an oscillating character and is periodic with the period

$$S^\alpha(\mathbf{X}) \rightarrow S^\alpha(\mathbf{X}) + 2\pi\epsilon$$

for each function  $S^\alpha(\mathbf{X})$ .

According to relations (4.6) and (4.11), the expansions (4.32) - (4.33) naturally generate the expansions depending on  $\Delta\boldsymbol{\theta}_0(\mathbf{X})$  for the brackets of the functionals  $\tilde{S}^{\alpha[\zeta]}(\mathbf{X})$  and  $S^{\alpha[\zeta]}(\mathbf{Y})$  on  $\mathcal{K}$ , which can be used in the region of definition of these functionals ( $|\mathbf{S}(\mathbf{X}) - \zeta(\mathbf{X})| = O(\epsilon)$ ). Thus, we can write:

$$\begin{aligned} \left\{ S^{\alpha[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}} &= \epsilon \left\{ S^{\alpha[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[0]}} + \epsilon^2 \left\{ S^{\alpha[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[1]}} + \dots \\ \left\{ S^{\alpha[\zeta]}(\mathbf{X}), S^{\beta[\zeta]}(\mathbf{Y}) \right\} \Big|_{\mathcal{K}} &= \epsilon^2 \left\{ S^{\alpha[\zeta]}(\mathbf{X}), S^{\beta[\zeta]}(\mathbf{Y}) \right\} \Big|_{\mathcal{K}^{[0]}} + \dots \end{aligned}$$

or

$$\left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}} = \epsilon \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[0]}} + \epsilon^2 \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}^{[1]}} + \dots \quad (4.34)$$

$$\left\{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \right\} \Big|_{\mathcal{K}} = \epsilon^2 \left\{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \right\} \Big|_{\mathcal{K}[0]} + \dots \quad (4.35)$$

All the terms above represent local expressions of  $\mathbf{S}_{\mathbf{X}}$ ,  $\zeta_{\mathbf{X}}$ ,  $\mathbf{U}(\mathbf{X})$ ,  $\mathbf{f}(\mathbf{X})$ ,  $\mathbf{h}(\mathbf{X})$ ,  $\mathbf{b}(\mathbf{X})$  and their derivatives with the additional dependence on  $\Delta\theta_0(\mathbf{X})$ , integrated over  $\mathbb{R}^d$ . Let us note also that the indices  $[k]$  mean here just the total number of differentiations of the functions  $\mathbf{S}_{\mathbf{X}}$ ,  $\zeta_{\mathbf{X}}$ ,  $\mathbf{U}(\mathbf{X})$ ,  $\mathbf{f}(\mathbf{X})$ ,  $\mathbf{h}(\mathbf{X})$ ,  $\mathbf{b}(\mathbf{X})$  w.r.t.  $\mathbf{X}$ . For  $\mathbf{S}(\mathbf{X}) \equiv \zeta(\mathbf{X})$  the relations above represent just the regular graded expansions in our previous sense.

The operator  $\delta/\delta S^\alpha(\mathbf{X})$  can be written here as the operator

$$\frac{\delta}{\delta S^\alpha(\mathbf{X})} = \frac{1}{\epsilon} \frac{\partial}{\partial \Delta\theta_0^\alpha} + \frac{\partial}{\partial X^q} \frac{\partial}{\partial S_{X^q}^\alpha} + \frac{\partial^2}{\partial X^q \partial X^p} \frac{\partial}{\partial S_{X^q X^p}^\alpha} + \dots \quad (4.36)$$

applied to the integrands in (4.34) - (4.35). According to relations (4.12) we can see that the first term in (4.36) can be actually omitted at the “points” of  $\mathcal{K}$  with  $\mathbf{S}(\mathbf{X}) \equiv \zeta(\mathbf{X})$ . As a result, we can actually claim that the values

$$\left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}}, \quad \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}}$$

are still represented by the regular graded expansions, and, besides that, we can write

$$\begin{aligned} \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} &= \epsilon \left[ \delta \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}[0]} / \delta S^\alpha(\mathbf{X}) \right] \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} + O(\epsilon^2) \\ \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \}}{\delta U^\gamma(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} &= \epsilon \left[ \delta \left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}[0]} / \delta U^\gamma(\mathbf{X}) \right] \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} + O(\epsilon^2) \\ \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} &= O(\epsilon^2), \quad \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, S_{[\mathbf{h}]}^{[\zeta]} \}}{\delta U^\gamma(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon^2) \end{aligned}$$

at the points  $(\mathbf{U}(\mathbf{X}), \zeta(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Using the same arguments we can claim also the analogous statements about the functionals, defined with the aid of the functionals  $S^{\alpha[\zeta]}(\mathbf{X})$ . Thus, using the definition (4.13), we can also write

$$\begin{aligned} \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]} \}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} &= \epsilon \left[ \delta \left\{ S_{[\mathbf{f}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]} \right\} \Big|_{\mathcal{K}[0]} / \delta S^\alpha(\mathbf{X}) \right] \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} + O(\epsilon^2) \\ \left. \frac{\delta \{ S_{[\mathbf{f}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]} \}}{\delta U^\gamma(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} &= \epsilon \left[ \delta \left\{ S_{[\mathbf{f}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]} \right\} \Big|_{\mathcal{K}[0]} / \delta U^\gamma(\mathbf{X}) \right] \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} + O(\epsilon^2) \end{aligned}$$

at the points  $(\mathbf{U}(\mathbf{X}), \zeta(\mathbf{X}))$  on  $\mathcal{K}$ .

Consider the Hamiltonian flow, generated by the functional  $J_{[\mathbf{b}]}$ , at the points of the submanifold  $\mathcal{K}$ :

$$\varphi(\theta, \mathbf{X}) = \Phi \left( \theta + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right)$$

It's not difficult to see, that in the main order of  $\epsilon$  the flow leaves invariant the submanifold  $\mathcal{K}$ , generating the linear evolution of the phase shifts  $\theta_0^\alpha(\mathbf{X})$  with the frequencies  $\omega^{\alpha\gamma}(\mathbf{X}) b_\gamma(\mathbf{X})$  such that we have

$$\left\{ \varphi^i(\boldsymbol{\theta}, \mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}} = \Phi_{\theta^\alpha}^i \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \omega^{\alpha\gamma}(\mathbf{X}) b_\gamma(\mathbf{X}) + O(\epsilon)$$

So, using the values of the functionals  $\mathbf{S}^{[\zeta]}(\mathbf{X})$  on  $\mathcal{K}$ , we can write:

$$\left\{ S^{\alpha[\zeta]}(\mathbf{X}), J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}} = \epsilon \omega^{\alpha\gamma}(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) b_\gamma(\mathbf{X}) + \dots$$

where the next corrections are represented by the terms containing the higher derivatives of the functions  $\mathbf{S}_{\mathbf{X}}, \boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}), \mathbf{f}(\mathbf{X}), \mathbf{b}(\mathbf{X})$ . We then immediately get:

$$\left\{ S_{[\mathbf{f}]}^{[\zeta]}, J_{[\mathbf{b}]} \right\} \Big|_{\mathcal{K}_{[0]}} = \int \omega^{\alpha\gamma}(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) f_\alpha(\mathbf{X}) b_\gamma(\mathbf{X}) d^d X$$

According to definition (4.13), we have also

$$\left\{ \varphi^i(\boldsymbol{\theta}, \mathbf{X}), U_{[\mathbf{b}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} = \Phi_{\theta^\alpha}^i \left( \boldsymbol{\theta} + \frac{\mathbf{S}(\mathbf{X})}{\epsilon}, \mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \omega^{\alpha\gamma}(\mathbf{X}) b_\gamma(\mathbf{X}) + O(\epsilon) \quad (4.37)$$

$$\left\{ S^{\alpha[\zeta]}(\mathbf{X}), U_{[\mathbf{b}]}^{[\zeta]} \right\} \Big|_{\mathcal{K}} = \epsilon \omega^{\alpha\gamma}(\mathbf{S}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) b_\gamma(\mathbf{X}) + \dots$$

At the same time we can write

$$\left\{ \varphi^i(\boldsymbol{\theta}, \mathbf{X}), S_{[\mathbf{h}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} = O(\epsilon) \quad (4.38)$$

Let us introduce now the unified set of functionals  $G^{\nu[\zeta]}(\mathbf{X})$ ,  $\nu = 1, \dots, 2m + s$ , putting

$$(G^{1[\zeta]}(\mathbf{X}), \dots, G^{2m+s[\zeta]}(\mathbf{X})) = (S^{1[\zeta]}(\mathbf{X}), \dots, S^{m[\zeta]}(\mathbf{X}), U^{1[\zeta]}(\mathbf{X}), \dots, U^{m+s[\zeta]}(\mathbf{X}))$$

and the unified notations for the coordinates  $(\mathbf{U}(\mathbf{X}), \mathbf{S}(\mathbf{X}))$  on  $\mathcal{K}$ :

$$(G^1(\mathbf{X}), \dots, G^{2m+s}(\mathbf{X})) = (S^1(\mathbf{X}), \dots, S^m(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X}))$$

We define also the regularized functionals

$$G_{[\mathbf{q}]}^{[\zeta]} = \int G^{\nu[\zeta]}(\mathbf{X}) q_\nu(\mathbf{X}) d^d X$$

(summation in  $\nu = 1, \dots, 2m + s$ ) for smooth compactly supported vector-valued functions  $\mathbf{q}(\mathbf{X}) = (q_1(\mathbf{X}), \dots, q_{2m+s}(\mathbf{X}))$ .

According to our considerations above, we have then in general case:

$$\left\{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} = O(\epsilon) \quad , \quad \frac{\delta \{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \}}{\delta G^\nu(\mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} = O(\epsilon)$$

Besides that, we can write

$$\left\{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(1)} = \left\{ G_{[\mathbf{q}]}, G_{[\mathbf{p}]} \right\}_{\text{AV}} \Big|_{[\mathbf{U}, \boldsymbol{\zeta}]} \quad (4.39)$$

$$\left. \frac{\delta\{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta G^\nu(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} = \left. \frac{\delta\{G_{[\mathbf{q}]}, G_{[\mathbf{p}]\}}_{\text{AV}}\}}{\delta G^\nu(\mathbf{X})} \right|_{[\mathbf{U}, \zeta]} \quad (4.40)$$

where the form  $\{\dots, \dots\}_{\text{AV}}$  on the space  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  is defined by relations (3.16).

Let us note that we use here the notation  $(k)$  to designate just the corresponding order of  $\epsilon$ . As we saw above, this order can be different from the gradation degree  $[k]$  for expressions containing the functionals  $\mathbf{S}^{[\zeta]}(\mathbf{X})$ .

According to the considerations above we can also write

$$\left. \frac{\delta\{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad (4.41)$$

where all the expressions above are given by the regular graded series at the point  $(\mathbf{U}(\mathbf{X}), \boldsymbol{\zeta}(\mathbf{X}))$ .

Using our remark about the flow, generated by the functional  $J_{[\mathbf{b}]}$ , we can easily get the relations

$$\left. \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), J_{[\mathbf{b}]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \left. \{g_{[\mathbf{Q}]}^{[\zeta]}, J_{[\mathbf{b}]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

and also

$$\left. \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), U_{[\mathbf{b}]}^{[\zeta]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \left. \{g_{[\mathbf{Q}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

However, we should write in general

$$\left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, J_{[\mathbf{b}]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(1) \quad , \quad \left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, U_{[\mathbf{b}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(1)$$

For the functionals  $S_{[\mathbf{f}]}^{[\zeta]}$  it's not difficult to get the relations

$$\left. \{g_{[\mathbf{Q}]}^{[\zeta]}, S_{[\mathbf{f}]}^{[\zeta]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, S_{[\mathbf{f}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

In general, we have to write for the regularized functionals  $G_{[\mathbf{q}]}^{[\zeta]}$ :

$$\left. \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), G_{[\mathbf{q}]}^{[\zeta]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \left. \{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]}\} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad (4.42)$$

$$\left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(1) \quad (4.43)$$

In the same way as before, we can state also for our functionals  $\mathbf{S}^{[\zeta]}(\mathbf{X})$ ,  $\mathbf{U}^{[\zeta]}(\mathbf{X})$ ,  $\mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$ , the relations:

$$\left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta S^\alpha(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta U^\gamma(\mathbf{X})} \right|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

or, in the “unified” form:

$$\left. \frac{\delta\{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{P}]}^{[\zeta]}\}}{\delta G^\nu(\mathbf{X})} \right|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}[\mathbf{U}, \zeta]} = O(\epsilon) \quad (4.44)$$

The pairwise Poisson brackets of the functionals  $g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X})$  can be represented by the following a little bit bulky expression

$$\begin{aligned} \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), g^{j[\zeta]}(\boldsymbol{\theta}', \mathbf{Y})\} &= \{\varphi^i(\boldsymbol{\theta}, \mathbf{X}), \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} - \\ &- \{\varphi^i(\boldsymbol{\theta}, \mathbf{X}), U^{\lambda[\zeta]}(\mathbf{Y})\} \Phi_{U^\lambda}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) - \\ &- \left\{ \varphi^i(\boldsymbol{\theta}, \mathbf{X}), S_{Y^p}^{\beta[\zeta]} \right\} \Phi_{k_p^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) - \\ &- \Phi_{U^\nu}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{U^{\nu[\zeta]}(\mathbf{X}), \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} - \\ &- \Phi_{k_q^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S_{X^q}^{\alpha[\zeta]}, \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} + \\ &+ \Phi_{U^\nu}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{U^{\nu[\zeta]}(\mathbf{X}), U^{\lambda[\zeta]}(\mathbf{Y})\} \Phi_{U^\lambda}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \Phi_{k_q^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S_{X^q}^{\alpha[\zeta]}, U^{\lambda[\zeta]}(\mathbf{Y})\} \Phi_{U^\lambda}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \Phi_{U^\nu}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{U^{\nu[\zeta]}(\mathbf{X}), S_{Y^p}^{\beta[\zeta]}\} \Phi_{k_p^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \Phi_{k_q^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S_{X^q}^{\alpha[\zeta]}, S_{Y^p}^{\beta[\zeta]}\} \Phi_{k_p^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) - \\ &- \frac{1}{\epsilon} \{\varphi^i(\boldsymbol{\theta}, \mathbf{X}), S^{\beta[\zeta]}(\mathbf{Y})\} \Phi_{\theta'^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) - \\ &- \frac{1}{\epsilon} \Phi_{\theta^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S^{\alpha[\zeta]}(\mathbf{X}), \varphi^j(\boldsymbol{\theta}', \mathbf{Y})\} + \\ &+ \frac{1}{\epsilon} \Phi_{\theta^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S^{\alpha[\zeta]}(\mathbf{X}), U^{\lambda[\zeta]}(\mathbf{Y})\} \Phi_{U^\lambda}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \frac{1}{\epsilon} \Phi_{\theta^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S^{\alpha[\zeta]}(\mathbf{X}), S_{Y^p}^{\beta[\zeta]}\} \Phi_{k_p^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \frac{1}{\epsilon} \Phi_{U^\nu}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{U^{\nu[\zeta]}(\mathbf{X}), S^{\beta[\zeta]}(\mathbf{Y})\} \Phi_{\theta'^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \frac{1}{\epsilon} \Phi_{k_q^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S_{X^q}^{\alpha[\zeta]}, S^{\beta[\zeta]}(\mathbf{Y})\} \Phi_{\theta'^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) + \\ &+ \frac{1}{\epsilon^2} \Phi_{\theta^\alpha}^i \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{X})}{\epsilon} + \boldsymbol{\theta}, \mathbf{X} \right) \{S^{\alpha[\zeta]}(\mathbf{X}), S^{\beta[\zeta]}(\mathbf{Y})\} \Phi_{\theta'^\beta}^j \left( \frac{\mathbf{S}^{[\zeta]}(\mathbf{Y})}{\epsilon} + \boldsymbol{\theta}', \mathbf{Y} \right) \end{aligned}$$

where we put  $\Phi(\theta, \mathbf{X}) \equiv \Phi(\theta, \mathbf{S}_{X^1}^{[\zeta]}, \dots, \mathbf{S}_{X^d}^{[\zeta]}, \mathbf{U}^{[\zeta]}(\mathbf{X}))$ .

However, using relations (4.37) and (4.17), it's not difficult to check that

$$\left\{ g_{[\mathbf{Q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]} \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} = \int Q_i(\theta, \mathbf{X}) \hat{B}_{[0]}^{ij}(\mathbf{X}) P_j(\theta, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} d^d X$$

where

$$\begin{aligned} \hat{B}_{[0]}^{ij}(\mathbf{X}) &= \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij}(\Phi(\theta, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})), \zeta_{X^1}^{\gamma_1} \Phi_{\theta^{\gamma_1}}, \dots, \zeta_{X^d}^{\gamma_d} \Phi_{\theta^{\gamma_d}}, \dots) \times \\ &\quad \times \zeta_{X^1}^{\alpha_1^1} \dots \zeta_{X^1}^{\alpha_{l_1}^1} \dots \zeta_{X^d}^{\alpha_d^1} \dots \zeta_{X^d}^{\alpha_{l_d}^d} \frac{\partial}{\partial \theta^{\alpha_1^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \dots \frac{\partial}{\partial \theta^{\alpha_d^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \end{aligned}$$

It's not difficult to check also the relations:

$$\frac{\delta \{g_{[\mathbf{Q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]}\}}{\delta G^\nu(\mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(1) \quad , \quad \frac{\delta \{g_{[\mathbf{Q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]}\}}{\delta \varphi^k(\theta, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(1) \quad (4.45)$$

Using relations (4.10), (4.18) and the invariance of the bracket (4.1) w.r.t. the transformations  $\theta^\alpha \rightarrow \theta^\alpha + \text{const}$ , we can also write for all the brackets  $\{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}$ ,  $\{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}$ ,  $\{g_{[\mathbf{Q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]}\}$  the relations:

$$\int \frac{\delta \{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^i(\theta, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} \Phi_{\theta^\alpha}^i \left( \theta + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0$$

(etc. ...).

The relations above take place in all orders of  $\epsilon$  and can in fact be strengthened in the main order. Indeed, we can see that the arbitrary functions  $q_\nu(\mathbf{X})$  appear in the main order of  $\epsilon$  just as local factors in the integrands of the above expressions. In the same way, making the change  $\tilde{Q}_i(\theta, \mathbf{W}) \rightarrow \tilde{Q}_i(\theta, \mathbf{W}) \mu_i(\mathbf{W})$  with arbitrary smooth functions  $\mu_i(\mathbf{W})$ , we get the same change  $Q_i(\theta, \mathbf{W}) \rightarrow Q_i(\theta, \mathbf{W}) \mu_i(\mathbf{W})$  for the functionals  $Q_i(\theta, \mathbf{W})$ . Here again the functions  $\mu_i(\mathbf{W})$  appear just as local factors  $\mu_i(\mathbf{X})$  in the main order of  $\epsilon$ . We can see then, that we can omit the integration w.r.t.  $\mathbf{X}$  in the main order of  $\epsilon$  and write for any  $\mathbf{X}$ :

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta \{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^i(\theta, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} \Phi_{\theta^\alpha}^i \left( \theta + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad (4.46)$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta \{g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^i(\theta, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \Phi_{\theta^\alpha}^i \left( \theta + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad (4.47)$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{\delta \{g_{[\mathbf{Q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]}\}}{\delta \varphi^i(\theta, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \Phi_{\theta^\alpha}^i \left( \theta + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right) \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad (4.48)$$

One of the main goals of this chapter is to prove that the form (3.16) gives in fact a Poisson bracket on the space of fields  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$ . For the justification of this fact the resolvability of the following systems

$$\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X}) B_{[f]}^I \left( \theta + \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X} \right) = - \left\{ g^{i[\zeta]}(\theta, \mathbf{X}), S_{[f]}^{[\zeta]} \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} \quad (4.49)$$



$$\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X}) B_{j[\mathbf{b}]}^{\Pi} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X} \right) = - \left\{ g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), U_{[\mathbf{b}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \quad (4.50)$$

where

$$\begin{aligned} \hat{B}_{[0][\zeta]}^{ij}(\mathbf{X}) \equiv & \sum_{l_1, \dots, l_d} B_{(l_1, \dots, l_d)}^{ij} \left( \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon}, \zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X}) \right), \dots \right) \times \\ & \times \zeta_{X^1}^{\alpha_1^1} \dots \zeta_{X^1}^{\alpha_{l_1}^1} \dots \zeta_{X^d}^{\alpha_1^d} \dots \zeta_{X^d}^{\alpha_{l_d}^d} \frac{\partial}{\partial \theta^{\alpha_1^1}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \dots \frac{\partial}{\partial \theta^{\alpha_1^d}} \dots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \end{aligned}$$

on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions will play important role.

Let us consider systems (4.49) - (4.50) in more detail under the assumptions that the family  $\Lambda$  represents a regular Hamiltonian submanifold equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let us note, that systems (4.49) - (4.50) are in fact analogous to system (2.29) considered in Chapter 2. Easy to see that systems (4.49) - (4.50) represent linear non-homogeneous differential systems in  $\boldsymbol{\theta}$  with periodic coefficients at every fixed value of  $\mathbf{X}$ . Systems (4.49) - (4.50) are independent for different  $\mathbf{X}$ , while the operators  $\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X})$  coincide with the corresponding operators  $\hat{B}_{\mathbf{k}_1, \dots, \mathbf{k}_d}^{ij}$  after the trivial shift:  $\boldsymbol{\theta} \rightarrow \boldsymbol{\theta} + \zeta(\mathbf{X})/\epsilon$ . We can also claim here, that all the “regular” kernel vectors of the operators  $\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X})$  on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions are given by the values

$$v_{i[\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})]}^{(k)} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon} \right) = \sum_{\gamma=1}^{m+s} \gamma_\gamma^k (\zeta_{X^1}, \dots, \zeta_{X^d}, \mathbf{U}(\mathbf{X})) \zeta_{i[\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})]}^{(\gamma)} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon} \right)$$

where the functions  $\zeta_{i[\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}]}^{(\gamma)}(\boldsymbol{\theta})$  are defined by relations (3.11).

Like in the case of system (2.29), we can formulate here the following lemma:

**Lemma 4.1.**

*Let the family  $\Lambda$  represent a regular Hamiltonian submanifold equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Then the right-hand parts of systems (4.49) - (4.50) are automatically orthogonal to the regular kernel vectors of the corresponding operators  $\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X})$ .*

*Proof.*

Indeed, view the relations (4.16) the convolution of the variation derivatives  $\delta U^{\gamma[\zeta]}(\mathbf{Z})/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  with the brackets  $\{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), S_{[\mathbf{f}]}^{[\zeta]}\}_{|\mathcal{K}}$  or  $\{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), U_{[\mathbf{b}]}^{[\zeta]}\}_{|\mathcal{K}}$  is identically equal to zero on  $\mathcal{K}$ . From the relations (4.13) we can see then that the values  $\delta U^{\gamma[\zeta]}(\mathbf{Z})/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  can be replaced in the leading order of  $\epsilon$  by the values  $\delta J^\gamma(\mathbf{Z})/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  at  $\boldsymbol{\varphi} = \boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}$ . Using the explicit expressions for the derivatives  $\delta J^\gamma(\mathbf{Z})/\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})$  at  $\boldsymbol{\varphi} = \boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}$ :

$$\left. \frac{\delta J^\gamma(\mathbf{Z})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}} = \sum_{l_1, \dots, l_d} \Pi_i^{\gamma(l_1 \dots l_d)} \left( \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Z})}{\epsilon}, \mathbf{Z} \right), \dots \right) \epsilon^{l_1 + \dots + l_d} \delta_{l_1 Z^1 \dots l_d Z^d}(\mathbf{Z} - \mathbf{X})$$

we can then write the corresponding convolutions as the action of the operator

$$\int_0^{2\pi} \dots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \sum_{l_1, \dots, l_d} \Pi_i^{\gamma(l_1 \dots l_d)} \left( \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{Z})}{\epsilon}, \mathbf{Z} \right), \dots \right) \epsilon^{l_1 + \dots + l_d} \frac{d^{l_1}}{dZ^{1 l_1}} \dots \frac{d^{l_d}}{dZ^{d l_d}}$$

on the functions  $\{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{Z}), S_{[\mathbf{f}]}^{[\zeta]}\}_{|\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}}$ ,  $\{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{Z}), U_{[\mathbf{b}]}^{[\zeta]}\}_{|\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}}$ . Easy to see that in the leading order of  $\epsilon$  the corresponding action is given by the action of the operator

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \sum_{l_1, \dots, l_d} \Pi_i^{\gamma(l_1 \dots l_d)} \left( \boldsymbol{\Phi} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{Z})}{\epsilon}, \mathbf{Z} \right), \dots \right) \times \\ \times \zeta_{Z^1}^{\alpha_1^1} \cdots \zeta_{Z^1}^{\alpha_{l_1}^1} \cdots \zeta_{Z^d}^{\alpha_1^d} \cdots \zeta_{Z^d}^{\alpha_{l_d}^d} \frac{\partial}{\partial \theta^{\alpha_1^1}} \cdots \frac{\partial}{\partial \theta^{\alpha_{l_1}^1}} \cdots \frac{\partial}{\partial \theta^{\alpha_1^d}} \cdots \frac{\partial}{\partial \theta^{\alpha_{l_d}^d}} \quad (4.51)$$

on the same functions.

Since the right-hand parts of systems (4.49) - (4.50) represent the leading order of the corresponding brackets at  $\boldsymbol{\varphi} = \boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}$ , we get that they should vanish under the action of the operator (4.51). After the integration by parts we then get immediately the orthogonality of the right-hand parts of (4.49) - (4.50) to the values  $\zeta_{i[\zeta \mathbf{x}, \mathbf{U}(\mathbf{X})]}^{(\gamma)}(\boldsymbol{\theta} + \boldsymbol{\zeta}(\mathbf{X})/\epsilon)$ .

Lemma 4.1 is proved.

In general, we can write

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \zeta_{i[\zeta \mathbf{x}, \mathbf{U}(\mathbf{X})]}^{(\gamma)} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon} \right) \left\{ g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), G_{[\mathbf{q}]}^{[\zeta]} \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} = 0 \quad (4.52)$$

Like in the case of system (2.29), we can also claim here that systems (4.49) - (4.50) are resolvable on the space of  $2\pi$ -periodic functions of  $\theta$  in the single-phase situation. However, the investigation of resolvability of systems (4.49) - (4.50) is much more complicated in the general multi-phase case, where the behavior of eigen-values of the operators  $\hat{B}_{[0][\zeta]}^{ij}(\mathbf{X})$  can be rather nontrivial. In general, we should expect the resolvability of systems (4.49) - (4.50) just on the subset  $\mathcal{S}'$  in the space of parameters

$$(\mathbf{k}_1(\mathbf{X}), \dots, \mathbf{k}_d(\mathbf{X}), \mathbf{U}(\mathbf{X})) = (\boldsymbol{\zeta}_{X^1}, \dots, \boldsymbol{\zeta}_{X^d}, \mathbf{U}(\mathbf{X}))$$

For the corresponding values of  $(\boldsymbol{\zeta}_{X^1}, \dots, \boldsymbol{\zeta}_{X^d}, \mathbf{U}(\mathbf{X}))$  we can then write in the general form

$$\left\{ G_{[\mathbf{q}]}^{[\zeta]}, g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} = \hat{B}_{[0][\zeta]}^{ij}(\mathbf{X}) B_{j[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right)$$

In the full analogy with (2.30) we can then also write here the relations

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{\theta^\alpha}^i \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \boldsymbol{\zeta}_{\mathbf{x}}, \mathbf{U}(\mathbf{X}) \right) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \frac{d^m \theta}{(2\pi)^m} = 0 \quad (4.53)$$

for the corresponding solutions of (4.49) - (4.50).

Let us denote here by  $\mathcal{M}'$  the subset in the space of parameters  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ , such that  $(\mathbf{k}_1, \dots, \mathbf{k}_d) \in \mathcal{M}$ . Easy to see that the set  $\mathcal{M}'$  has the full measure in the space  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ .

Let us recall also that the notation  $\Lambda$  denotes the family of  $m$ -phase solutions of (2.2) given by formula (2.5). As before, the notation  $\hat{\Lambda}$  represents the corresponding set of  $2\pi$ -periodic in each  $\theta^\alpha$  functions  $\boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$ , satisfying system (2.3).

#### Theorem 4.1.

Let the family  $\Lambda$  represent a regular Hamiltonian submanifold equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let the parameter space  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$  of the family  $\Lambda$  have a dense set  $\mathcal{S}' \subset \mathcal{M}'$  on which systems (4.49) - (4.50) are resolvable on the space of smooth

$2\pi$ -periodic in each  $\theta^\alpha$  functions. Then the relations (3.16) define a Poisson bracket on the space of fields  $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$ .

Proof.

The skew-symmetry of the form (3.16) easily follows from the skew-symmetry of the bracket (4.1). Let us prove the Jacobi identity for the form (3.16). Without loss of generality, we will give here the proof for the point  $\mathbf{G}(\mathbf{X}) = (\boldsymbol{\zeta}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Let us fix now the functions  $\mathbf{q}(\mathbf{X})$ ,  $\mathbf{p}(\mathbf{X})$  and the functional  $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$  and consider the Jacobi identity of the form

$$\left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \left\{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \right\} \right\} + \left\{ G_{[\mathbf{p}]}^{[\zeta]}, \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \right\} \right\} + \left\{ G_{[\mathbf{q}]}^{[\zeta]}, \left\{ G_{[\mathbf{p}]}^{[\zeta]}, g_{[\mathbf{Q}]}^{[\zeta]} \right\} \right\} \equiv 0 \quad (4.54)$$

Using expansions (4.19) - (4.20) and relations (4.41), (4.42) - (4.44), it is not difficult to see that the leading term ( $\sim \epsilon$ ) of relation (4.54) at  $\boldsymbol{\varphi} = \boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}$  can be written as

$$\begin{aligned} & \int \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} d^d X + \\ & + \int \left\{ G_{[\mathbf{p}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} d^d X - \\ & - \int \left\{ G_{[\mathbf{q}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0 \end{aligned}$$

We can write again the above identity in a stronger form. Thus, making again the change  $\tilde{Q}_i(\boldsymbol{\theta}, \mathbf{X}) \rightarrow \tilde{Q}_i(\boldsymbol{\theta}, \mathbf{X}) \mu_i(\mathbf{X})$ , we can change again the functionals  $Q_i(\boldsymbol{\theta}, \mathbf{X})$ :  $Q_i(\boldsymbol{\theta}, \mathbf{X}) \rightarrow Q_i(\boldsymbol{\theta}, \mathbf{X}) \mu_i(\mathbf{X})$  by arbitrary smooth multipliers  $\mu_i(\mathbf{X})$ . Easy to see also that the functions  $\mu_i(\mathbf{X})$  appear then just as simple local multipliers in the integrands in the zero order of  $\epsilon$ . By the arbitrariness of  $\mu_i(\mathbf{X})$ , we can then write again for every  $\mathbf{X}$ :

$$\begin{aligned} & \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} + \\ & + \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ G_{[\mathbf{p}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} - \\ & - \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ G_{[\mathbf{q}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} \equiv 0 \end{aligned}$$

Using relations (4.17) we can write

$$\begin{aligned} & \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}} = \\ & = \int Q_j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \left\{ g^{j[\zeta]}(\boldsymbol{\theta}', \mathbf{W}), \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}} \frac{d^m \theta'}{(2\pi)^m} d^d W = \end{aligned}$$

$$\begin{aligned}
&= \int Q_j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \left\{ \varphi^j(\boldsymbol{\theta}', \mathbf{W}), \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} d^d W - \\
&- \int Q_j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \Phi_{k_q^\alpha}^j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \times \\
&\quad \times \left\{ S_{W_q^\alpha}^{\alpha[\boldsymbol{\zeta}]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} d^d W - \\
&- \int Q_j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \Phi_{U^\gamma}^j \left( \boldsymbol{\theta}' + \frac{\boldsymbol{\zeta}(\mathbf{W})}{\epsilon}, \mathbf{W} \right) \times \\
&\quad \times \left\{ U^{\gamma[\boldsymbol{\zeta}]}(\mathbf{W}), \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}} \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} d^d W
\end{aligned}$$

Thus, we have in the main order of  $\epsilon$ :

$$\begin{aligned}
\left\{ g_{[\mathbf{Q}]}^{[\boldsymbol{\zeta}]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(0)} &= -\hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) - \\
&- \int_0^{2\pi} \dots \int_0^{2\pi} Q_j(\boldsymbol{\theta}', \mathbf{X}) \Phi_{U^\gamma}^j(\boldsymbol{\theta}', \mathbf{X}) \frac{d^m \boldsymbol{\theta}'}{(2\pi)^m} \cdot \omega^{\alpha\gamma}(\mathbf{X}) \Phi_{\theta^\alpha}^k \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right)
\end{aligned}$$

(view the skew-symmetry of the operator  $\hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X})$ ).

Using relations (4.46), we have then

$$\begin{aligned}
\int_0^{2\pi} \dots \int_0^{2\pi} \left\{ g_{[\mathbf{Q}]}^{[\boldsymbol{\zeta}]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(0)} \frac{\delta \{G_{[\mathbf{q}]}^{[\boldsymbol{\zeta}]}, G_{[\mathbf{p}]}^{[\boldsymbol{\zeta}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(1)} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} &= \\
= - \int_0^{2\pi} \dots \int_0^{2\pi} \left[ \hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{G_{[\mathbf{q}]}^{[\boldsymbol{\zeta}]}, G_{[\mathbf{p}]}^{[\boldsymbol{\zeta}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(1)} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m}
\end{aligned}$$

Now assume that  $(\boldsymbol{\zeta}_\mathbf{X}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ , so we can write the relations

$$\left\{ G_{[\mathbf{q}]}^{[\boldsymbol{\zeta}]}, g^{k[\boldsymbol{\zeta}]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(1)} = \hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) B_{j[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \quad (4.55)$$

where  $B_{j[\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X})$  represent some smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions of the variables  $\boldsymbol{\theta}$ .

We can write then at the corresponding  $\mathbf{X} \in \mathbb{R}^d$ :

$$\begin{aligned}
&\int_0^{2\pi} \dots \int_0^{2\pi} \left[ \hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{G_{[\mathbf{q}]}^{[\boldsymbol{\zeta}]}, G_{[\mathbf{p}]}^{[\boldsymbol{\zeta}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(1)} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} - \\
&- \int_0^{2\pi} \dots \int_0^{2\pi} \left[ \hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) B_{j[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{g_{[\mathbf{Q}]}^{[\boldsymbol{\zeta}]}, G_{[\mathbf{q}]}^{[\boldsymbol{\zeta}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(0)} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} + \\
&+ \int_0^{2\pi} \dots \int_0^{2\pi} \left[ \hat{B}_{[0][\boldsymbol{\zeta}]}^{kj}(\mathbf{X}) B_{j[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{g_{[\mathbf{Q}]}^{[\boldsymbol{\zeta}]}, G_{[\mathbf{p}]}^{[\boldsymbol{\zeta}]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \boldsymbol{\zeta}]}(0)} \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0 \quad (4.56)
\end{aligned}$$

Let us consider now the Jacobi identity of the form

$$\left\{ g_{[\mathbf{P}]}^{[\zeta]}, \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \right\} \right\} + \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \left\{ G_{[\mathbf{q}]}^{[\zeta]}, g_{[\mathbf{P}]}^{[\zeta]} \right\} \right\} + \left\{ G_{[\mathbf{q}]}^{[\zeta]}, \left\{ g_{[\mathbf{P}]}^{[\zeta]}, g_{[\mathbf{Q}]}^{[\zeta]} \right\} \right\} \equiv 0 \quad (4.57)$$

where the functionals  $\mathbf{P}(\boldsymbol{\theta}, \mathbf{X})$  and  $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$  are defined with the aid of arbitrary functions  $\tilde{\mathbf{P}}(\boldsymbol{\theta}, \mathbf{X})$ ,  $\tilde{\mathbf{Q}}(\boldsymbol{\theta}, \mathbf{X})$ .

According to relations (4.37), (4.38) and (4.48), we can write then in the main order of  $\epsilon$ :

$$\begin{aligned} & \int \left\{ g_{[\mathbf{P}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} d^d X - \\ & - \int \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ g_{[\mathbf{P}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} d^d X \equiv 0 \end{aligned}$$

Having the functions  $q_\nu(\mathbf{X})$  just as simple local factors in the integrands, we can write again the above relation in the stronger form:

$$\begin{aligned} & \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ g_{[\mathbf{P}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} - \\ & - \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ g_{[\mathbf{Q}]}^{[\zeta]}, \varphi^k(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{\delta \{ g_{[\mathbf{P}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} \equiv 0 \end{aligned}$$

Using the same calculations as before with the relations (4.47), we can rewrite the above relations in the form:

$$\begin{aligned} & \int_0^{2\pi} \cdots \int_0^{2\pi} \left[ \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) P_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} - \\ & - \int_0^{2\pi} \cdots \int_0^{2\pi} \left[ \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{\delta \{ g_{[\mathbf{P}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} \frac{d^m \theta}{(2\pi)^m} \equiv 0 \end{aligned}$$

It is easy to see that the values

$$\frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)}$$

can be represented in the form:

$$\frac{\delta \{ g_{[\mathbf{Q}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(0)} = q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) Q_i \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right)$$

where  $\hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X})$  is a linear operator on the space of smooth periodic functions of  $\boldsymbol{\theta}$ .

The above relations can then be finally written as:

$$\begin{aligned}
& \int_0^{2\pi} \cdots \int_0^{2\pi} \left[ \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) P_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
& \quad \times \left[ q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) Q_i \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{d^m \theta}{(2\pi)^m} - \\
& - \int_0^{2\pi} \cdots \int_0^{2\pi} \left[ \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
& \quad \times \left[ q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) P_i \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad (4.58)
\end{aligned}$$

Let us note now that the values  $\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X})$  and  $\mathbf{P}(\boldsymbol{\theta}, \mathbf{X})$  represent here arbitrary smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions, satisfying conditions (4.17). In particular, for  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$  we can substitute the values

$$\mathbf{P}(\boldsymbol{\theta}, \mathbf{X}) = \mathbf{B}_{[\mathbf{p}]}(\boldsymbol{\theta}, \mathbf{X}) \quad \text{or} \quad \mathbf{P}(\boldsymbol{\theta}, \mathbf{X}) = \mathbf{B}_{[\mathbf{q}]}(\boldsymbol{\theta}, \mathbf{X}) \quad (4.59)$$

in the identity (4.58).

As a result, for  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$  we can rewrite relations (4.56) in the form:

$$\begin{aligned}
& \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
& \quad \times \left( \frac{\delta \{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \bigg|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} - q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) + \right. \\
& \quad \left. + p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right) \equiv 0
\end{aligned}$$

Using the skew-symmetry of the operator  $\hat{B}_{[0][\zeta]}^{kj}(\mathbf{X})$  we can then write under the same conditions

$$\begin{aligned}
& \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} Q_j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \times \\
& \quad \times \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) \left( \frac{\delta \{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta \varphi^k(\boldsymbol{\theta}, \mathbf{X})} \bigg|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} - q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) + \right. \\
& \quad \left. + p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right) \equiv 0
\end{aligned}$$

The values  $Q_j(\boldsymbol{\theta}, \mathbf{X})$  represent here arbitrary smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions satisfying the restriction (4.17). At the same time the values in the brackets represent some smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions of  $\boldsymbol{\theta}$  for  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ . As a corollary, we can then write for some values

$a_{[\mathbf{q}, \mathbf{p}]}^\alpha(\mathbf{X})$ :

$$\begin{aligned} \hat{B}_{[0][\zeta]}^{kj}(\mathbf{X}) \left( \frac{\delta\{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{X})} \right) \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} - q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) + \\ + p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \Big) \equiv \sum_{\alpha=1}^m a_{[\mathbf{q}, \mathbf{p}]}^\alpha(\mathbf{X}) \Phi_{\theta^\alpha}^j \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \end{aligned}$$

at the corresponding  $\mathbf{X} \in \mathbb{R}^d$ .

We can see that the values in parentheses represent again some smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions of  $\boldsymbol{\theta}$  at the same point  $\mathbf{X} \in \mathbb{R}^d$ . At the same time, the right-hand part of the relation above represents a linear combination of the flows, generating linear evolution of the phase shifts  $\theta_0^\alpha$  with the coefficients  $a_{[\mathbf{q}, \mathbf{p}]}^\alpha(\mathbf{X})$ . For a regular Hamiltonian submanifold  $\Lambda$  with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$  we can conclude then that the covector in parentheses is given by some linear combination of the functions  $\zeta_{[\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})]}^{(\gamma)}(\boldsymbol{\theta} + \boldsymbol{\zeta}(\mathbf{X})/\epsilon)$ , modulo the kernel vectors of the operator  $\hat{B}_{[0][\zeta]}^{kj}(\mathbf{X})$ . For  $(\zeta_{X^1}, \dots, \zeta_{X^d}) \in \mathcal{M}$  all the smooth in  $\boldsymbol{\theta}$  kernel vectors of  $\hat{B}_{[0][\zeta]}^{kj}(\mathbf{X})$  on the space of  $2\pi$ -periodic in each  $\theta^\alpha$  functions are given by the regular kernel vectors, which are also represented by linear combinations of  $\zeta_{[\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})]}^{(\gamma)}(\boldsymbol{\theta} + \boldsymbol{\zeta}(\mathbf{X})/\epsilon)$ . Thus, we can write for  $(\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ :

$$\begin{aligned} \frac{\delta\{G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} - q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) + \\ + p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \equiv \sum_{\gamma=1}^{m+s} b_{\gamma[\mathbf{q}, \mathbf{p}]}(\mathbf{X}) \zeta_{[\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})]}^{(\gamma)} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \quad (4.60) \end{aligned}$$

with some coefficients  $b_{\gamma[\mathbf{q}, \mathbf{p}]}(\mathbf{X})$ .

Now, let us consider the Jacobi identity of the form

$$\left\{ G_{[\mathbf{q}]}^{[\zeta]}, \left\{ G_{[\mathbf{p}]}^{[\zeta]}, G_{[\mathbf{r}]}^{[\zeta]} \right\} \right\} + \left\{ G_{[\mathbf{p}]}^{[\zeta]}, \left\{ G_{[\mathbf{r}]}^{[\zeta]}, G_{[\mathbf{q}]}^{[\zeta]} \right\} \right\} + \left\{ G_{[\mathbf{r}]}^{[\zeta]}, \left\{ G_{[\mathbf{q}]}^{[\zeta]}, G_{[\mathbf{p}]}^{[\zeta]} \right\} \right\} \equiv 0$$

for arbitrary smooth functions  $\mathbf{q}(\mathbf{X})$ ,  $\mathbf{p}(\mathbf{X})$ ,  $\mathbf{r}(\mathbf{X})$ .

In the leading ( $\sim \epsilon^2$ ) order of  $\epsilon$  at  $\boldsymbol{\varphi} = \boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}$  the above identity gives the relations:

$$\begin{aligned} \int \left\{ G_{[\mathbf{q}]}^{[\zeta]}, G^{\nu[\zeta]}(\mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta\{G_{[\mathbf{p}]}^{[\zeta]}, G_{[\mathbf{r}]}^{[\zeta]}\}}{\delta G^\nu(\mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} d^d X + c.p. + \quad (4.61) \\ + \int \left\{ G_{[\mathbf{q}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta\{G_{[\mathbf{p}]}^{[\zeta]}, G_{[\mathbf{r}]}^{[\zeta]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} d^d X + c.p. \equiv 0 \end{aligned}$$

Using relations (4.52) and representations (4.55), (4.60), we can write for  $(\zeta_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ :

$$\int_0^{2\pi} \dots \int_0^{2\pi} \left\{ G_{[\mathbf{q}]}^{[\zeta]}, g^{k[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta\{G_{[\mathbf{p}]}^{[\zeta]}, G_{[\mathbf{r}]}^{[\zeta]}\}}{\delta\varphi^k(\boldsymbol{\theta}, \mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} + c.p. = \quad (4.62)$$

$$\begin{aligned}
&= \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0]|\zeta}^{kj}(\mathbf{X}) B_{j[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
&\quad \times \left[ p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{r}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) - r_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] + \\
&+ \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0]|\zeta}^{kj}(\mathbf{X}) B_{j[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
&\quad \times \left[ r_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) - q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{r}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] + \\
&+ \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0]|\zeta}^{kj}(\mathbf{X}) B_{j[\mathbf{r}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
&\quad \times \left[ q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) - p_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{q}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right]
\end{aligned}$$

Making now the substitutions

$$\mathbf{Q}(\boldsymbol{\theta}, \mathbf{X}) = \mathbf{B}_{[\mathbf{r}]}(\boldsymbol{\theta}, \mathbf{X}) \quad , \quad \mathbf{P}(\boldsymbol{\theta}, \mathbf{X}) = \mathbf{B}_{[\mathbf{p}]}(\boldsymbol{\theta}, \mathbf{X})$$

in identity (4.58), we get the following identities

$$\begin{aligned}
&\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0]|\zeta}^{kj}(\mathbf{X}) B_{j[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
&\quad \times \left[ q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{r}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] - \\
&- \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d^m \theta}{(2\pi)^m} \left[ \hat{B}_{[0]|\zeta}^{kj}(\mathbf{X}) B_{j[\mathbf{r}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \times \\
&\quad \times \left[ q_\nu(\mathbf{X}) \hat{K}_{k[\mathbf{U}, \zeta]}^{\nu i}(\mathbf{X}) B_{i[\mathbf{p}]} \left( \boldsymbol{\theta} + \frac{\boldsymbol{\zeta}(\mathbf{X})}{\epsilon}, \mathbf{X} \right) \right] \equiv 0
\end{aligned}$$

for  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ .

Making the cyclic permutations of the functions  $\mathbf{q}(\mathbf{X})$ ,  $\mathbf{p}(\mathbf{X})$ , and  $\mathbf{r}(\mathbf{X})$  in the above identity, we can see then that the right-hand part of the equality (4.62) is identically equal to zero for all values  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X})) \in \mathcal{S}'$ . At the same time we can see that the left-hand side of relation (4.62) represents a smooth regular function of the values  $(\boldsymbol{\zeta}_{\mathbf{X}}, \mathbf{U}(\mathbf{X}))$  and their derivatives. From the fact that the set  $\mathcal{S}'$  represents an everywhere dense set in the parameter space, we can then conclude that the left-hand side of the relation (4.62) is identically equal to zero under the conditions of the theorem.

From the identity (4.61) we get then the relations:

$$\int \left\{ G_{[\mathbf{q}]}^{[\zeta]}, G^{\nu[\zeta]}(\mathbf{X}) \right\} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} \frac{\delta \{G_{[\mathbf{p}]}^{[\zeta]}, G_{[\mathbf{r}]}^{[\zeta]}\}}{\delta G^{\nu}(\mathbf{X})} \Big|_{\boldsymbol{\varphi}=\boldsymbol{\Phi}_{[\mathbf{U}, \zeta]}(1)} d^d X + c.p. \equiv 0$$



Using relations (4.39) - (4.40) we get now the Jacobi identity for the bracket  $\{\dots, \dots\}_{AV}$  at the “point”  $(\zeta(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Theorem 4.1 is proved.

According to our remarks above, we can formulate here the following theorem about the single-phase case:

**Theorem 4.1'.**

*Let the family  $\Lambda$  of single-phase solutions of (2.2) represent a regular Hamiltonian submanifold in the space of periodic functions equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{s+1})$ . Then the form*

$$\begin{aligned} \{S(\mathbf{X}), S(\mathbf{Y})\} &= 0, \\ \{S(\mathbf{X}), U^\gamma(\mathbf{Y})\} &= \omega^\gamma(S_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{s+1}(\mathbf{X})) \delta(\mathbf{X} - \mathbf{Y}), \\ \{U^\gamma(\mathbf{X}), U^\rho(\mathbf{Y})\} &= \langle A_{10\dots 0}^{\gamma\rho} \rangle(S_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{s+1}(\mathbf{X})) \delta_{X^1}(\mathbf{X} - \mathbf{Y}) + \dots + \\ &+ \langle A_{0\dots 01}^{\gamma\rho} \rangle(S_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{s+1}(\mathbf{X})) \delta_{X^d}(\mathbf{X} - \mathbf{Y}) + \\ &+ [\langle Q^{\gamma\rho p} \rangle(S_{\mathbf{X}}, U^1(\mathbf{X}), \dots, U^{s+1}(\mathbf{X}))]_{X^p} \delta(\mathbf{X} - \mathbf{Y}), \quad \gamma, \rho = 1, \dots, s+1 \end{aligned}$$

*defines a Poisson bracket on the space of fields  $(S(\mathbf{X}), U^\gamma(\mathbf{X}))$ ,  $\gamma = 1, \dots, s+1$ .*

The following theorem proves the invariance of the (contravariant) 2-form (3.16) with respect to the choice of the functionals  $(I^1, \dots, I^{m+s})$ .

**Theorem 4.2.**

*Let the family  $\Lambda$  represent a regular Hamiltonian submanifold equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Let the set  $(I'^1, \dots, I'^{m+s})$  represent another set of commuting integrals, satisfying all the conditions of Definition 2.2. Then the forms (3.16), obtained with the aid of the sets  $(I^1, \dots, I^{m+s})$  and  $(I'^1, \dots, I'^{m+s})$  coincide with each other.*

Proof.

Let us consider two different sets of parameters on  $\Lambda$ :

$$(\mathbf{k}_1, \dots, \mathbf{k}_d, U^1, \dots, U^{m+s}), \quad (\mathbf{k}_1, \dots, \mathbf{k}_d, U'^1, \dots, U'^{m+s})$$

where  $U^\gamma \equiv \langle P^\gamma \rangle$ ,  $U'^\gamma \equiv \langle P'^\gamma \rangle$ . We can then write on  $\Lambda$ :

$$U'^\gamma = U'^\gamma(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

since the values  $(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$  give the full set of parameters on  $\Lambda$  excluding the initial phase shifts.

We have to prove here that the brackets (3.16)  $\{\dots, \dots\}_{AV}$  and  $\{\dots, \dots\}'_{AV}$ , obtained with the aid of the sets  $(I^1, \dots, I^{m+s})$  and  $(I'^1, \dots, I'^{m+s})$ , transform into each other under the corresponding transformation

$$U'^\gamma(\mathbf{X}) = U'^\gamma(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}(\mathbf{X}))$$

We have by definition

$$\{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\}_{AV} = 0, \quad \{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\}'_{AV} = 0$$

in both the brackets  $\{\dots, \dots\}_{\text{AV}}$  and  $\{\dots, \dots\}'_{\text{AV}}$ . Let us consider now the brackets, containing the functionals  $\mathbf{U}(\mathbf{X})$  and  $\mathbf{U}'(\mathbf{X})$ .

We can write on  $\Lambda$ :

$$U^\gamma \equiv \langle P^\gamma \rangle \equiv J^\gamma|_\Lambda, \quad U'^\gamma \equiv \langle P'^\gamma \rangle \equiv J'^\gamma|_\Lambda \quad (4.63)$$

where the functionals  $J^\gamma$  and  $J'^\gamma$  are defined by formula (2.15).

By Definition 2.2, both the sets  $(J^1, \dots, J^{m+s})$ ,  $(J'^1, \dots, J'^{m+s})$  generate the linear shifts of  $\boldsymbol{\theta}_0$  on  $\Lambda$  according to bracket (2.16), such that we have

$$\text{rk } \|\omega^{\alpha\gamma}\| = m, \quad \text{rk } \|\omega'^{\alpha\gamma}\| = m$$

for the corresponding sets  $\{\omega^\gamma\}$ ,  $\{\omega'^\gamma\}$ .

Besides that, the variation derivatives

$$\zeta_i^\gamma(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) = \left. \frac{\delta J^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}}, \quad \zeta_i'^\gamma(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) = \left. \frac{\delta J'^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}}$$

represent regular covectors on  $\hat{\Lambda}$ , such that both the linear spans

$$\text{Span } \{\zeta^1, \dots, \zeta^{m+s}\}, \quad \text{Span } \{\zeta'^1, \dots, \zeta'^{m+s}\}$$

contain all the regular kernel vectors of the Hamiltonian operators (2.11). As then follows from Definition 2.1, the linear spans of the sets  $\{\zeta^1, \dots, \zeta^{m+s}\}$  and  $\{\zeta'^1, \dots, \zeta'^{m+s}\}$  coincide at every point of the submanifold  $\hat{\Lambda}$ . According to (4.63), we can then write

$$\left. \frac{\delta J'^\gamma}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}} = \frac{\partial U'^\gamma}{\partial U^\rho}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) \left. \frac{\delta J^\rho}{\delta \varphi^i(\boldsymbol{\theta})} \right|_{\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}}$$

for any submanifold  $\hat{\Lambda}_{\mathbf{k}_1, \dots, \mathbf{k}_d}$ .

From the relations above we also immediately get the relations

$$\omega'^{\alpha\gamma}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U}) = \frac{\partial U'^\gamma}{\partial U^\rho} \omega^{\alpha\rho}(\mathbf{k}_1, \dots, \mathbf{k}_d, \mathbf{U})$$

for the corresponding frequencies  $\{\omega^\gamma\}$  and  $\{\omega'^\gamma\}$ .

Using the relations

$$\delta U'^\gamma(\mathbf{Y}) = \frac{\partial U'^\gamma}{\partial U^\rho}(\mathbf{Y}) \delta U^\rho(\mathbf{Y}) + \frac{\partial U'^\gamma}{\partial k_q^\alpha}(\mathbf{Y}) \delta S_{Y^q}^\alpha$$

and relations (3.16) we get then:

$$\begin{aligned} \{S^\alpha(\mathbf{X}), U'^\gamma(\mathbf{Y})\}_{\text{AV}} &= \frac{\partial U'^\gamma}{\partial U^\rho}(\mathbf{X}) \omega^{\alpha\rho}(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y}) = \\ &= \omega'^{\alpha\gamma}(\mathbf{X}) \delta(\mathbf{X} - \mathbf{Y}) = \{S^\alpha(\mathbf{X}), U'^\gamma(\mathbf{Y})\}'_{\text{AV}} \end{aligned}$$

To finish the proof of the Theorem let us recall, that we have the relations

$$\{U^\gamma(\mathbf{X}), U^\rho(\mathbf{Y})\}_{\text{AV}} = \{J^\gamma(\mathbf{X}), J^\rho(\mathbf{Y})\}|_{\mathcal{K}(1)}$$

$$\{U'^\gamma(\mathbf{X}), U'^\rho(\mathbf{Y})\}'_{\text{AV}} = \{J'^\gamma(\mathbf{X}), J'^\rho(\mathbf{Y})\}|_{\mathcal{K}(1)}$$

where the functionals  $J^\gamma(\mathbf{X})$ ,  $J'^\gamma(\mathbf{X})$  are defined by relations (4.3). Consider now the regularized functionals

$$J'_{[\mathbf{a}]} = \int a_\gamma(\mathbf{Z}) J'^\gamma(\mathbf{Z}) d^d Z$$

with some smooth compactly supported functions  $a_\gamma(\mathbf{Z})$ .

Using relations (4.20) we can write on  $\mathcal{K}$ :

$$\begin{aligned} \delta J'_{[\mathbf{a}]} &= \int a_\gamma(\mathbf{Z}) \left. \frac{\delta J'^\gamma(\mathbf{Z})}{\delta G^\nu(\mathbf{X})} \right|_{\mathcal{K}} \delta G^{\nu[\zeta]}(\mathbf{X}) d^d X d^d Z + \\ &\quad + \int a_\gamma(\mathbf{Z}) \left. \frac{\delta J'^\gamma(\mathbf{Z})}{\delta \varphi^i(\boldsymbol{\theta}, \mathbf{X})} \right|_{\mathcal{K}} \delta g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} d^d X d^d Z \end{aligned}$$

where the “coordinate system”  $(\mathbf{G}^{[\zeta]}(\mathbf{X}), \mathbf{g}^{[\zeta]}(\boldsymbol{\theta}, \mathbf{X}))$  is introduced with the aid of the functionals  $\mathbf{J}(\mathbf{X})$ .

Without loss of generality we will consider all the relations below at the “point”  $\mathbf{G}(\mathbf{X}) = (\zeta(\mathbf{X}), \mathbf{U}(\mathbf{X}))$  of the submanifold  $\mathcal{K}$ .

Expanding the values of the variation derivatives in the integrands it is not difficult to get the following relations:

$$\begin{aligned} \delta J'_{[\mathbf{a}]} &= \int a_\gamma(\mathbf{Z}) \left. \frac{\delta U'^\gamma(\mathbf{Z})}{\delta G^\nu(\mathbf{X})} \right|_{[\mathbf{U}, \zeta]} \delta G^{\nu[\zeta]}(\mathbf{X}) d^d X d^d Z + \int \epsilon U_{\nu[\mathbf{a}, \mathbf{G}]}(\mathbf{X}, \epsilon) \delta G^{\nu[\zeta]}(\mathbf{X}) d^d X + \\ &\quad + \int \left( a_\gamma(\mathbf{X}) \zeta_{i[\zeta \mathbf{x}, \mathbf{U}(\mathbf{X})]}'^\gamma \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon} \right) + \epsilon U_{i[\mathbf{a}, \mathbf{G}]} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon}, \mathbf{X}, \epsilon \right) \right) \delta g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}) \frac{d^m \theta}{(2\pi)^m} d^d X \end{aligned}$$

where  $U_{\nu[\mathbf{a}, \mathbf{G}]}(\mathbf{X}, \epsilon)$  and  $U_{i[\mathbf{a}, \mathbf{G}]}(\boldsymbol{\theta}, \mathbf{X}, \epsilon)$  are local functions of  $(\mathbf{a}(\mathbf{X}), \zeta \mathbf{x}, \mathbf{U}(\mathbf{X}))$  and their derivatives, given by regular at  $\epsilon \rightarrow 0$  series in  $\epsilon$ . Easy to see also, that we have the relations

$$\frac{\delta U'^\gamma(\mathbf{Z})}{\delta U^\rho(\mathbf{X})} = \frac{\partial U'^\gamma}{\partial U^\rho}(\mathbf{Z}) \delta(\mathbf{Z} - \mathbf{X}) \quad , \quad \frac{\delta U'^\gamma(\mathbf{Z})}{\delta S^\alpha(\mathbf{X})} = \frac{\partial U'^\gamma}{\partial k_q^\alpha}(\mathbf{Z}) \delta_{Z^q}(\mathbf{Z} - \mathbf{X})$$

According to the requirements of the Theorem, the flows, generated by  $I'^\gamma$ , leave invariant the submanifold  $\Lambda$  and the values of  $U^\gamma$  on it. This property conserves in the main order of  $\epsilon$  for the functionals  $J'_{[\mathbf{a}]}$  and the submanifold  $\mathcal{K}$  with the coordinates  $U^\gamma(\mathbf{X})$ . So, we can write here

$$\{U^{\gamma[\zeta]}(\mathbf{X}), J'_{[\mathbf{b}]}\}|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon) \quad , \quad \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), J'_{[\mathbf{b}]}\}|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

We have also the relations

$$\{S^{\alpha[\zeta]}(\mathbf{X}), J'_{[\mathbf{b}]}\}|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}} = O(\epsilon)$$

according to our definition of the functionals  $S^{\alpha[\zeta]}(\mathbf{X})$ .

Besides that, in the full analogy with (4.52) we can write the relations

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \zeta_{i[\zeta \mathbf{x}, \mathbf{U}(\mathbf{X})]}^{(\rho)} \left( \boldsymbol{\theta} + \frac{\zeta(\mathbf{X})}{\epsilon} \right) \{g^{i[\zeta]}(\boldsymbol{\theta}, \mathbf{X}), J'_{[\mathbf{b}]}\}|_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} \frac{d^m \theta}{(2\pi)^m} = 0$$

for the constraints, defined with the aid of the functionals  $\mathbf{J}(\mathbf{X})$ .

Using the relations

$$\zeta'_{i[\zeta_{\mathbf{x}}, \mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta}) = \left. \frac{\partial U'^\gamma}{\partial U^\rho} \right|_{[\mathbf{U}, \zeta]} \zeta^{(\rho)}_{i[\zeta_{\mathbf{x}}, \mathbf{U}(\mathbf{X})]}(\boldsymbol{\theta})$$

we can then write

$$\{J'_{[\mathbf{a}]}, J'_{[\mathbf{b}]}\}_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} = \int a_\gamma(\mathbf{Z}) \left. \frac{\delta U'^\gamma(\mathbf{Z})}{\delta G^\nu(\mathbf{X})} \right|_{[\mathbf{U}, \zeta]} \{G^{\nu[\zeta]}(\mathbf{X}), J'_{[\mathbf{b}]}\}_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} d^d X d^d Z$$

Easy to see, that the same considerations can be repeated now also for the functional  $J'_{[\mathbf{b}]}$ . Finally, we can write

$$\begin{aligned} \{U'_{[\mathbf{a}]}, U'_{[\mathbf{b}]}\}'_{\text{AV}} \Big|_{[\mathbf{U}, \zeta]} &= \{J'_{[\mathbf{a}]}, J'_{[\mathbf{b}]}\}_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} = \\ &= \int a_\gamma(\mathbf{Z}) \left. \frac{\delta U'^\gamma(\mathbf{Z})}{\delta G^\nu(\mathbf{X})} \right|_{[\mathbf{U}, \zeta]} \{G^{\nu[\zeta]}(\mathbf{X}), G^{\mu[\zeta]}(\mathbf{Y})\}_{\varphi=\Phi_{[\mathbf{U}, \zeta]}(1)} \left. \frac{\delta U'^\rho(\mathbf{W})}{\delta G^\mu(\mathbf{Y})} \right|_{[\mathbf{U}, \zeta]} b_\rho(\mathbf{W}) \times \\ &\quad \times d^d X d^d Y d^d Z d^d W = \\ &= \{U'_{[\mathbf{a}]}, U'_{[\mathbf{b}]}\}'_{\text{AV}} \Big|_{[\mathbf{U}, \zeta]} \end{aligned}$$

which completes the proof of the Theorem.

Theorem 4.2 is proved.

At last, let us consider the regular Whitham system (3.12). We will assume now that the family  $\Lambda$  represents a complete regular Hamiltonian family of  $m$ -phase solutions of system (2.2) according to Definition 3.1.

**Theorem 4.3.**

*Let  $\Lambda$  represent a complete regular Hamiltonian family of  $m$ -phase solutions of system (2.2) equipped with a minimal set of commuting integrals  $(I^1, \dots, I^{m+s})$ . Then the corresponding regular Whitham system (3.12) can be represented in the form*

$$S_T^\alpha = \{S^\alpha(\mathbf{X}), H_{av}\}_{\text{AV}}, \quad U_T^\gamma = \{U^\gamma(\mathbf{X}), H_{av}\}_{\text{AV}} \quad (4.64)$$

where the functional  $H_{av}$  is defined by formula (3.17).

Proof.

According to Theorem 4.2, we can assume without loss of generality that the Hamiltonian functional  $H$  is included in the set  $(I^1, \dots, I^{m+s})$ . In this case it can be easily seen that system (4.64) coincides with (3.12).

Theorem 4.3 is proved.

Let us formulate here also two theorems about the “canonical forms” of the bracket (3.16), which were formulated in [41] with a brief sketch of the proof.

**Theorem 4.4 ([41]).**

*Consider the Poisson bracket given by relations (3.16) with  $\text{rk} ||\omega^{\alpha\gamma}|| = m$ . There exists locally an invertible coordinate transformation*

$$\begin{aligned} (S^1(\mathbf{X}), \dots, S^m(\mathbf{X}), U^1(\mathbf{X}), \dots, U^{m+s}(\mathbf{X})) &\rightarrow \\ &\rightarrow (S^1(\mathbf{X}), \dots, S^m(\mathbf{X}), Q_1(\mathbf{X}), \dots, Q_m(\mathbf{X}), N^1(\mathbf{X}), \dots, N^s(\mathbf{X})) \end{aligned}$$

where

$$Q_\alpha = Q_\alpha(\mathbf{S}_\mathbf{X}, U^1, \dots, U^{m+s}) \quad , \quad N^l = N^l(\mathbf{S}_\mathbf{X}, U^1, \dots, U^{m+s}) \quad ,$$

taking the bracket (3.16) to the form:

$$\begin{aligned} \{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\} &= 0 \quad , \\ \{S^\alpha(\mathbf{X}), Q_\beta(\mathbf{Y})\} &= \delta_\beta^\alpha \delta(\mathbf{X} - \mathbf{Y}) \quad , \quad \{S^\alpha(\mathbf{X}), N^l(\mathbf{Y})\} = 0 \quad , \\ \{Q_\alpha(\mathbf{X}), Q_\beta(\mathbf{Y})\} &= J_{\alpha\beta}[\mathbf{S}, \mathbf{N}](\mathbf{X}, \mathbf{Y}) \quad , \\ \{Q_\alpha(\mathbf{X}), N^l(\mathbf{Y})\} &= J_\alpha^l[\mathbf{S}, \mathbf{N}](\mathbf{X}, \mathbf{Y}) \quad , \\ \{N^l(\mathbf{X}), N^q(\mathbf{Y})\} &= J^{lq}[\mathbf{S}, \mathbf{N}](\mathbf{X}, \mathbf{Y}) \quad , \end{aligned}$$

where the functionals  $J_{\alpha\beta}$ ,  $J_\alpha^l$ ,  $J^{lq}$  are given by general local distributions of the gradation degree 1.

For the special case  $s = 0$  Theorem 4.4 can be formulated in the stronger form:

**Theorem 4.5 ([41]).**

Consider the Poisson bracket given by relations (3.16) with  $\text{rk}||\omega^{\alpha\gamma}|| = m$  and  $s = 0$ . There exists locally an invertible coordinate transformation

$$(S^1(\mathbf{X}), \dots, S^m(\mathbf{X}), U^1(\mathbf{X}), \dots, U^m(\mathbf{X})) \rightarrow (S^1(\mathbf{X}), \dots, S^m(\mathbf{X}), Q_1(\mathbf{X}), \dots, Q_m(\mathbf{X}))$$

where  $Q_\alpha = Q_\alpha(\mathbf{S}_\mathbf{X}, \mathbf{U})$ , which takes the bracket (3.16) to the following non-degenerate canonical form:

$$\begin{aligned} \{S^\alpha(\mathbf{X}), S^\beta(\mathbf{Y})\} &= 0 \quad , \quad \{Q_\alpha(\mathbf{X}), Q_\beta(\mathbf{Y})\} = 0 \quad , \\ \{S^\alpha(\mathbf{X}), Q_\beta(\mathbf{Y})\} &= \delta_\beta^\alpha \delta(\mathbf{X} - \mathbf{Y}) \quad . \end{aligned}$$

As we can see, the averaged bracket has the simplest structure in the absence of additional parameters  $(n^1, \dots, n^s)$ . Let us say, that the presence of the parameters  $(n^1, \dots, n^s)$  can really make the structure of the bracket (3.16) more complicated. Let us note also, that both the theorems above are based just on the form of the bracket (3.16) and are not connected with the averaging procedure itself.

## 5 Illustrations: KdV and other examples.

Let us continue now with the example of the KdV equation and consider the construction of the Poisson brackets for the corresponding Whitham systems according to our scheme.

Let us consider again the regular Hamiltonian submanifolds  $\Lambda^{(m)}$  defined by the sets of equations (2.42). The corresponding system (2.3) can be written here in the form

$$\omega^\alpha(\mathbf{U}) \Phi_{\theta^\alpha} = k^\alpha(\mathbf{U}) \Phi \Phi_{\theta^\alpha} - k^\alpha(\mathbf{U}) k^\beta(\mathbf{U}) k^\gamma(\mathbf{U}) \Phi_{\theta^\alpha \theta^\beta \theta^\gamma} \quad (5.1)$$

Let us say here that the eigenmodes of the linearized operator (5.1), as well as the adjoint operator, were studied in detail ([29, 6, 7, 30, 31]). In particular, we can claim here that all the families  $\Lambda^{(m)}$  represent complete regular Hamiltonian families of  $m$ -phase solutions of KdV equipped with minimal sets of commuting integrals  $(I^1, \dots, I^{m+1})$ . It can be also shown, that the minimal set of commuting integrals for  $\Lambda^{(m)}$  can be given in fact by any  $m + 1$  independent higher integrals  $(I^{j_1}, \dots, I^{j_{m+1}})$  of the KdV equation.

It is well-known that the Whitham systems of the KdV hierarchy represent a famous object in the theory of integrable systems and were investigated in different aspect in many details. The Hamiltonian structures of the Whitham systems for KdV were considered in the frame of the general Hamiltonian theory of the Whitham equations started in the works of B.A. Dubrovin and S.P. Novikov ([16, 17, 18, 19]). Let us say, that the Hamiltonian theory of the Whitham equations gives a basis for the integrability of the Whitham hierarchies corresponding to a wide class of integrable systems. Usually, the Whitham hierarchies are considered there as systems of Hydrodynamic Type.

As we said already, we consider here the Whitham system in the form (3.12) and try to construct the corresponding Hamiltonian structure, having the form (3.16) - (3.17).

The KdV equation gives a very convenient example demonstrating the applicability of Theorem 4.1 in the multi-phase case. Indeed, we have to require here the resolvability of the multi-phase systems (4.49) - (4.50) on a dense set  $\mathcal{S}' \subset \mathcal{M}'$  in the space of parameters on  $\Lambda^{(m)}$ , which represents in general a nontrivial condition. However, the requirements of Theorem 4.1 can be easily established here, using the same approach which was used in Chapter 2 for the construction of the finite-dimensional bracket on the family  $\Lambda^{(m)}$ .

We note first that the operator  $\hat{B}_{[0]}(X)$  is given now in the form

$$\hat{B}_{[0]}(X) = k^1(X) \frac{\partial}{\partial \theta^1} + \dots + k^m(X) \frac{\partial}{\partial \theta^m}$$

The regular zero mode of the operator  $\hat{B}_{[0]}(X)$  is given by the constant function on the torus  $\mathbb{T}^m$  and is orthogonal to the right-hand parts of systems (4.49) - (4.50) according to Lemma 4.1. The resolvability of systems (4.49) - (4.50) on a dense set  $\mathcal{S}' \subset \mathcal{M}'$  is based then on the analytic properties of the right-hand parts of (4.49) - (4.50).

Indeed, it is not difficult to get again from the theta-functional representation of the  $m$ -phase solutions of KdV that the Fourier components in  $\boldsymbol{\theta}$  of the right-hand parts of systems (4.49) - (4.50) decay faster than any power of  $|\mathbf{n}|$  ( $\mathbf{n} = (n_1, \dots, n_m)$ ) for any fixed  $\mathbf{X}$ . At the same time, the zero Fourier components ( $\mathbf{n} = (0, \dots, 0)$ ) of the right-hand parts of systems (4.49) - (4.50) are identically equal to zero according to Lemma 4.1. As a result, we can claim again that systems (4.49) - (4.50) can be resolved on the space of smooth  $2\pi$ -periodic in each  $\theta^\alpha$  functions for all Diophantine vectors  $\mathbf{k} = (k^1, \dots, k^m)$  with index  $\nu > m - 1$ . Using again the fact, that the set of these vectors has the full measure in the space  $(k^1, \dots, k^m)$ , we can apply Theorem 4.1 in our situation.

As another example, consider now the one-dimensional local Poisson bracket

$$\{\varphi(x), \varphi(y)\} = \delta'''(x - y) + \beta^2 \delta'(x - y) \quad (5.2)$$

and the Hamiltonian functional

$$H = \int \varphi^3 dx$$

The corresponding evolution system has the form

$$\varphi_t = 6\varphi\varphi_{xxx} + 18\varphi_x\varphi_{xx} + 6\beta^2\varphi\varphi_x \quad (5.3)$$

The bracket (5.2) has one local translationally invariant annihilator, having the form

$$N = \int \varphi dx$$

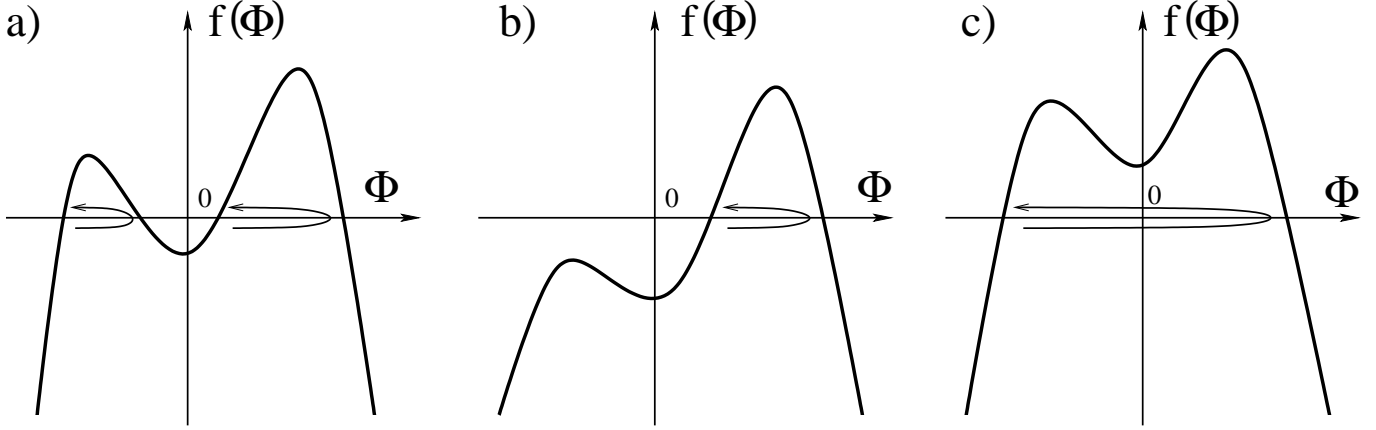


Figure 1: Different periodic cycles, corresponding to different regions in the space of parameters  $(\omega, k, A, B)$ .

The periodic one-phase solutions of (5.3) are defined from the equation

$$\omega \Phi_{\theta} = 6k^3 \Phi \Phi_{\theta\theta\theta} + 18k^3 \Phi_{\theta} \Phi_{\theta\theta} + 6\beta^2 k \Phi \Phi_{\theta} \quad (5.4)$$

which can be integrated to the form

$$\frac{\omega}{3} \Phi^3 - 3k^3 \Phi^2 \Phi_{\theta}^2 - \frac{3}{4} \beta^2 k \Phi^4 - A \Phi^2 - B = 0$$

( $A = \text{const}$ ,  $B = \text{const}$ ). We have then

$$k \frac{\Phi d\Phi}{\sqrt{\omega \Phi^3/9k - \beta^2 \Phi^4/4 - A \Phi^2/3k - B/3k}} = \pm d\theta$$

so, the function  $\Phi(\theta + \theta_0)$  can be expressed in terms of elliptic functions. The corresponding dependence between  $\Phi$  and  $\theta$  is given by the integration with respect to  $\Phi$  over the periodic cycles, restricting the areas of positive values of the function

$$f(\Phi) = \omega \Phi^3/9k - \beta^2 \Phi^4/4 - A \Phi^2/3k - B/3k$$

Under different choice of the parameters  $(\omega, k, A, B)$  the cycles can have different form shown at Fig. 1 (a, b, c). So, we can consider in fact several families of the one-phase solutions of system (5.3), corresponding to the cycles of different geometry. Let us say, however, that the Whitham systems for different families  $\Lambda$  can demonstrate rather different properties, which can restrict in fact the applicability of the Whitham theory in concrete problems. Thus, the regular Whitham system can have hyperbolic or elliptic nature depending on the type of the roots of the polynomial  $f(\Phi)$  (see e.g. [10]). The last case corresponds in fact to the modulation instability of the solutions of the corresponding type, so, the applicability of the Whitham approach should be specially studied in every concrete case. We will not study here these questions and consider just the Hamiltonian structure of the regular Whitham system.

The parameters  $(\omega, k, A, B)$  are connected by the relation

$$k \oint \frac{\Phi d\Phi}{\sqrt{\omega \Phi^3/9k - \beta^2 \Phi^4/4 - A \Phi^2/3k - B/3k}} = 2\pi$$

so, the family of one-phase solutions of (5.3) is parametrized by 3 independent parameters excluding the initial phase shift  $\theta_0$ . In particular, the total set of parameters on the full family  $\Lambda$  of one-phase solutions can be chosen in the form  $(k, U^1, U^2, \theta_0)$ , where

$$U^1 \equiv \langle \varphi \rangle = \int_0^{2\pi} \Phi(\theta) \frac{d\theta}{2\pi} \quad , \quad U^2 \equiv \langle \varphi^3 \rangle = \int_0^{2\pi} \Phi^3(\theta) \frac{d\theta}{2\pi}$$

are the values of the functionals  $N$  and  $H$  on the corresponding solutions of (5.3). It is easy to check that every submanifold  $\Lambda_k$  represents a regular Hamiltonian submanifold in the space of periodic functions with the period  $2\pi/k$  for all the values of  $k \neq \pm\beta/n$ ,  $n \in \mathbb{N}$ . It's not difficult to check also, that for the same values  $k \neq \pm\beta/n$  the total family  $\Lambda$  represents a complete Hamiltonian family of one-phase solutions of (5.3), equipped with the minimal set of commuting integrals  $(I^1, I^2) = (N, H)$ .

Easy to check that the conservation laws for the functionals  $N$  and  $H$  have the form

$$\varphi_t = (6\varphi\varphi_{xx} + 6\varphi_x^2 + 3\beta^2\varphi^2)_x \quad , \quad (\varphi^3)_t = \left(18\varphi^3\varphi_{xx} + \frac{9}{2}\beta^2\varphi^4\right)_x$$

so the regular Whitham system for the family  $\Lambda$  can be written as

$$\begin{aligned} S_T &= \omega(S_X, U^1, U^2) \\ U_T^1 &= 3\beta^2 \langle \Phi^2 \rangle_X \\ U_T^2 &= \left( -54 S_X^2 \langle \Phi^2 \Phi_\theta^2 \rangle + \frac{9}{2} \beta^2 \langle \Phi^4 \rangle \right)_X \end{aligned} \tag{5.5}$$

where all the values  $\langle \dots \rangle$  are represented as functions of  $(S_X, U^1, U^2)$ .

Calculation of the pairwise Poisson brackets of the densities of  $N$  and  $H$  gives the relations:

$$\{P_N(x), P_N(y)\} = \delta'''(x-y) + \beta^2 \delta'(x-y) \quad ,$$

$$\begin{aligned} \{P_N(x), P_H(y)\} &= 3\varphi^2(x) \delta'''(x-y) + 18\varphi\varphi_x \delta''(x-y) + \\ &+ 18(\varphi\varphi_{xx} + \varphi_x^2) \delta'(x-y) + 3\beta^2\varphi^2(x) \delta'(x-y) + \\ &+ 6(\varphi\varphi_{xxx} + 3\varphi_x\varphi_{xx}) \delta(x-y) + 6\beta^2\varphi\varphi_x \delta(x-y) \quad , \end{aligned}$$

$$\{P_H(x), P_N(y)\} = 3\varphi^2(x) \delta'''(x-y) + 6\beta^2\varphi^2(x) \delta'(x-y) \quad ,$$

$$\begin{aligned} \{P_H(x), P_H(y)\} &= 9\varphi^4(x) \delta'''(x-y) + 54\varphi^3\varphi_x \delta''(x-y) + \\ &+ 54(\varphi^3\varphi_{xx} + \varphi^2\varphi_x^2) \delta'(x-y) + 9\beta^2\varphi^4(x) \delta'(x-y) + \\ &+ 18(\varphi^3\varphi_{xxx} + 3\varphi^2\varphi_x\varphi_{xx}) \delta(x-y) + 18\beta^2\varphi^3\varphi_x \delta(x-y) \quad , \end{aligned}$$

Using Theorem 4.1' for the set  $k \neq \pm\beta/n$  and Theorem 4.3, we can claim now that the Whitham system (5.5) is Hamiltonian with respect to the bracket

$$\begin{aligned} \{S(X), S(Y)\} &= 0 \quad , \quad \{S(X), U^1(Y)\} = 0 \quad , \\ \{S(X), U^2(Y)\} &= \omega(S_X, U^1, U^2) \delta(X-Y) \quad , \end{aligned} \tag{5.6}$$



$$\begin{aligned}
\{U^1(X), U^1(Y)\} &= \beta^2 \delta'(X - Y) , \\
\{U^1(X), U^2(Y)\} &= 3\beta^2 \langle \Phi^2 \rangle \delta'(X - Y) + 3\beta^2 \langle \Phi^2 \rangle_X \delta(X - Y) , \\
\{U^2(X), U^1(Y)\} &= 3\beta^2 \langle \Phi^2 \rangle \delta'(X - Y) , \\
\{U^2(X), U^2(Y)\} &= (-108 S_X^2 \langle \Phi^2 \Phi_\theta^2 \rangle + 9\beta^2 \langle \Phi^4 \rangle) \delta'(X - Y) + \\
&+ (-54 S_X^2 \langle \Phi^2 \Phi_\theta^2 \rangle + 9\beta^2 \langle \Phi^4 \rangle / 2)_X \delta(X - Y)
\end{aligned}$$

with the Hamiltonian functional

$$H_{av} = \int \langle \Phi^3 \rangle dX$$

We will not make here further investigation of the example, let us just say, that the construction above can be easily generalized to the multi-dimensional case. Thus, using the Poisson bracket

$$\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\} = \sum_{q,p,r=1}^d \alpha_{qpr} \delta_{x^q x^p x^r}(\mathbf{x} - \mathbf{y}) + \sum_{q=1}^d \beta_q \delta_{x^q}(\mathbf{x} - \mathbf{y})$$

and the same Hamiltonian functional

$$H = \int \varphi^3 d^d x$$

we get different multi-dimensional analogs of system (5.3).

The corresponding one-phase solutions are defined in this case by equation equivalent to (5.4), which can be easily integrated in the same way. The full families of the one-phase solutions are parametrized in this case by the values  $(k_1, \dots, k_d, U^1, U^2, \theta_0)$  and represent complete Hamiltonian families equipped with a minimal set of commuting integrals  $(I^1, I^2)$  on the open set of the full measure in the space of  $(k_1, \dots, k_d)$ . As a result, we can suggest then a complete analog of the bracket (5.6) for the corresponding regular Whitham systems in the  $d$ -dimensional space.

Let us say again that we have chosen the above example as one of the simplest examples where the averaging of a Poisson bracket is made in presence of just a minimal set of commuting integrals in accordance with the idea of the paper.

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